Tropical Polynomials–Solutions

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1 Tropical Arithmetic

In tropical arithmetic, we define new addition and multiplication operations on the real numbers. The **tropical sum** of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

while the **tropical product** of two numbers is their sum:

$$x \odot y = x + y.$$

1. Which of the following properties hold in tropical arithmetic?

- **Addition is commutative:** $x \oplus y = y \oplus x$.

> True. $\min(\min(x, y), z) = \min(x, y, z) = \min(x, \min(y, z))$

- **Addition is associative:** $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

> True. $\min(x, y) = \min(y, x)$

- **An additive identity exists:** There exists a real number n such that $x \oplus n = x$ for all real numbers x.

> False. Such an n would satisfy $\min(x, n) = x$ or, equivalently, $x \leq n$, for all real numbers x.

2. Let's expand our number set to include a tropical additive identity. What would be an appropriate name for this new "number"? Give appropriate definitions for the tropical sum and tropical product of this new number with a general real number x and with itself.

> Because the tropical additive identity must be greater than or equal to every real number, we call it infinity (∞). For any real number x, we define
>
> $$\infty \oplus x = x$$
> $$\infty \oplus \infty = \infty$$
> $$\infty \odot x = \infty$$
> $$\infty \odot \infty = \infty$$

3. Which of the following properties hold in tropical arithmetic?

- **Additive inverses exist:** For each number x, there exists a number y such that $x \oplus y = n$, where n is the additive identity.

False. Unless $x = \infty$, there is no y such that $x \oplus y = \infty$, i.e., such that $\min(x, y) = \infty$.

- **Multiplication is associative:** $(x \odot y) \odot z = x \odot (y \odot z)$.

True. $(x + y) + z = x + (y + z)$

- **Multiplication is commutative:** $x \odot y = y \odot x$.

True. $x + y = y + x$

- **There exists a multiplicative identity:** There exists a number i such that $x \odot i = x$ for all numbers x.

True. The multiplicative identity is 0: $x \odot 0 = x + 0 = x$.

- **Multiplicative inverses exist:** For each number x not equal to the additive identity, there exists a number y such that $x \odot y = i$, where i is the multiplicative identity.

True. For $x \neq \infty$, $x \odot (-x) = x + (-x) = 0$.

- **Multiplication distributes over addition:** $x \odot (y \oplus z) = x \odot y \oplus x \odot z$.

True. $x + \min(y, z) = \min(x + y, x + z)$

4. Complete the tropical addition and multiplication tables below.

$\oplus$	1	2	3	4	∞
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
∞	1	2	3	4	∞

$\odot$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

5. Expand and simplify $f(x) = (x \oplus 2)(x \oplus 3)$, where juxtaposition represents tropical multiplication. Then use your expansion to compute $f(1)$ and $f(4)$.

$$\begin{aligned}(x \oplus 2)(x \oplus 3) &= x^2 \oplus 2x \oplus 3x \oplus (2 \odot 3)\\ &= x^2 \oplus (2 \oplus 3)x \oplus (2 \odot 3)\\ &= x^2 \oplus 2x \oplus 5\end{aligned}$$

$$\begin{aligned}f(1) &= 1^2 \oplus (2 \odot 1) \oplus 5\\ &= 2 \oplus 3 \oplus 5\\ &= 2\end{aligned}$$

$$\begin{aligned}f(4) &= 4^2 \oplus (2 \odot 4) \oplus 5\\ &= 8 \oplus 6 \oplus 5\\ &= 5\end{aligned}$$

2 Tropical Polynomials

A **polynomial** is an expression formed by adding and/or multiplying together numbers and copies of a variable x. Every polynomial can be written in the form

$$a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

for some nonnegative integer n and **coefficients** $a_n, \ldots, a_2, a_1, a_0$.

It follows from the **Fundamental Theorem of Algebra** that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with **real coefficients**. For example,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x^2 + 2x + 5)(x - 2)(x + 4)^2.$$

Over the complex numbers, any such polynomial can be factored completely into polynomials of degree 1 with **complex coefficients**. For the example above,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x + 1 - 2i)(x + 1 + 2i)(x - 2)(x + 4)^2.$$

The factors can be determined by computing the **roots** (or the "zeros") of the polynomial. The polynomial above has roots

$$-1 + 2i, -1 - 2i, 2, -4, -4.$$

We say that the root -4 has **multiplicity** 2.

There is a quadratic formula for determining the roots of a polynomial of degree 2, along with cubic and quartic formulas for degrees 3 and 4. However, starting with degree 5, there is no longer a nice formula which enables us to find the roots of every polynomial. For polynomials of large degree, we generally must settle for approximate roots, found by a computer.

A **tropical polynomial** is an expression formed by (tropically) adding and/or multiplying tropical numbers (i.e., real numbers or ∞) and copies of a variable x. Every tropical polynomial can be written in the form

$$(a_n \odot x^n) \oplus \cdots \oplus (a_2 \odot x^2) \oplus (a_1 \odot x) \oplus (a_0)$$

for some nonnegative integer n and **coefficients** $a_n, \ldots, a_2, a_1, a_0$. (Note that the exponents here represent repeated *tropical* multiplication.) For convenience, we represent tropical multiplication by juxtaposition, in the usual manner:

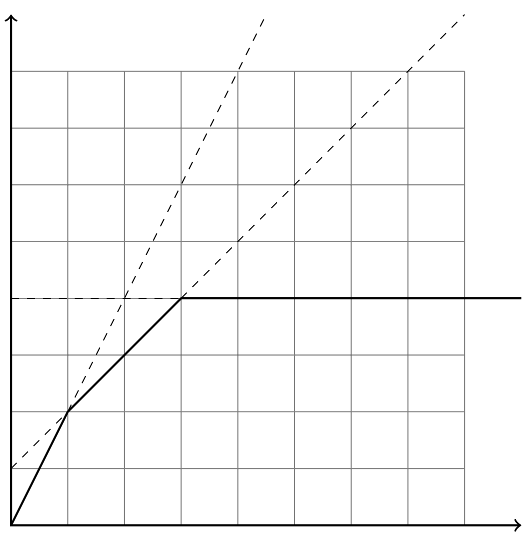
$$a_n x^n \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0.$$

Questions:

- *Can tropical polynomials always be factored completely into polynomials of degree 1?*
- *Is there a tropical quadratic formula for finding the roots of quadratic polynomials? How about a cubic formula?*
- *For polynomials of large degree, must we rely on a computer to find roots, or can we do it by hand?*

2.1 Tropical quadratic polynomials

6. Draw a precise graph of the tropical polynomial $f(x) = x^2 \oplus 1x \oplus 4$. You may find it helpful to first rewrite the tropical polynomial (as an expression involving standard operations) using the definitions of $\oplus$ and $\odot$.



In standard notation,

$$f(x) = \min(2x, 1 + x, 4).$$

Now, try to factor the tropical polynomial $x^2 \oplus 1x \oplus 4$ into linear (degree 1) factors. In other words, find numbers r and s such that

$$x^2 \oplus 1x \oplus 4 = (x \oplus r)(x \oplus s).$$

These numbers r and s are called the **roots** of the tropical polynomial. (Note that we use $x \oplus r$ and $x \oplus s$ because we do not have a tropical subtraction.)

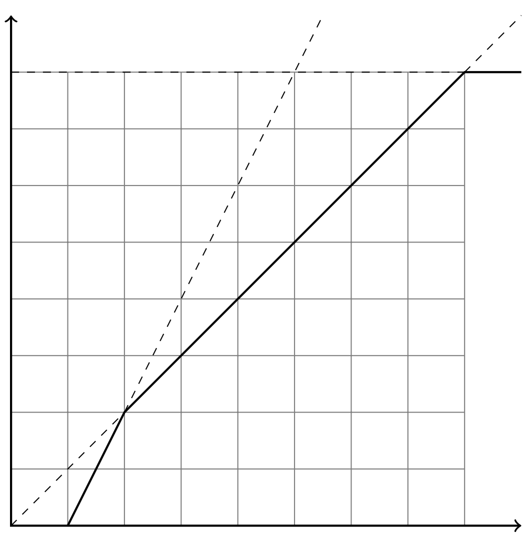
Because $(x \oplus r)(x \oplus s) = x^2 \oplus (r \oplus s)x \oplus rs$, we must have $r \oplus s = 1$ and $r \odot s = 4$. In standard notation, we need $\min(r, s) = 1$ and $r + s = 4$. We take $r = 1$ and $s = 3$:

$$f(x) = x^2 \oplus 1x \oplus 4 = (x \oplus 1)(x \oplus 3).$$

Do you notice any relationship between the graph and the factorization? Can you see the roots in the graph?

The roots 1 and 3 are the x-coordinates of the corners of the graph.

7. Graph $f(x) = -2x^2 \oplus x \oplus 8$, and then find a factorization of $f(x)$ in the form $a(x \oplus r)(x \oplus s)$. Can you see the roots r and s in the graph? How are the roots related to the coefficients of $f(x)$?



We (tropically) factor out a -2 to obtain

$$f(x) = -2(x^2 \oplus 2x \oplus 10).$$

Proceeding as in the previous problem, we obtain

$$f(x) = -2(x \oplus 2)(x \oplus 8).$$

The roots 2 and 8 are once again the x-coordinates of the corners of the graph. The roots are also the differences between consecutive coefficients of $f(x)$:

$$0 - (-2) = 2$$
$$8 - 0 = 8$$

8. Find a tropical polynomial $f(x)$ with a value of 7 for all sufficiently large x and with roots 4 and 5.

We are looking for $f(x) = ax^2 \oplus bx \oplus c$. We need $f(\infty) = 7$, so the constant term $c = 7$. In view of the pattern discovered above, we subtract 5 from the value of c to obtain $b = 2$, and we subtract 4 from the value of b to obtain $a = -2$. We conclude that

$$f(x) = -2x^2 \oplus 2x \oplus 7.$$

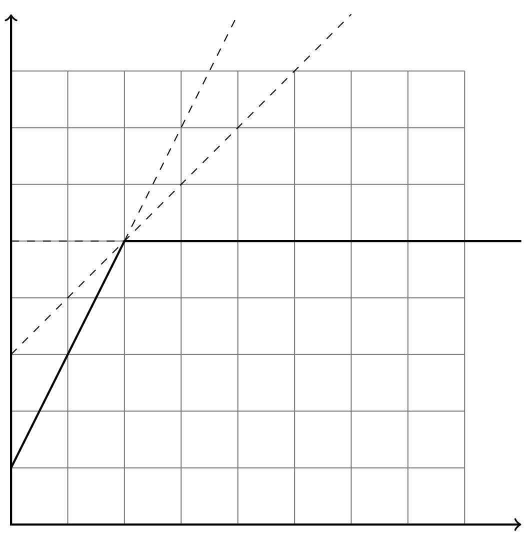
Note that it does not work to subtract the roots in the other order. Indeed,

$$-2x^2 \oplus 3x \oplus 7 \neq -2(x \oplus 4)(x \oplus 5).$$

The polynomial $-2x^2 \oplus 3x \oplus 7$ does not factor, although it defines the same function as another polynomial which does factor:

$$-2(x \oplus 4.5)^2 = -2x^2 \oplus 2.5x \oplus 7.$$

9. Graph $f(x) = 1x^2 \oplus 3x \oplus 5$, and then find a factorization in the form $f(x) = a(x \oplus r)(x \oplus s)$. How is this graph different from the previous ones? How is this factorization different from the others? How are the roots related to the coefficients of $f(x)$?



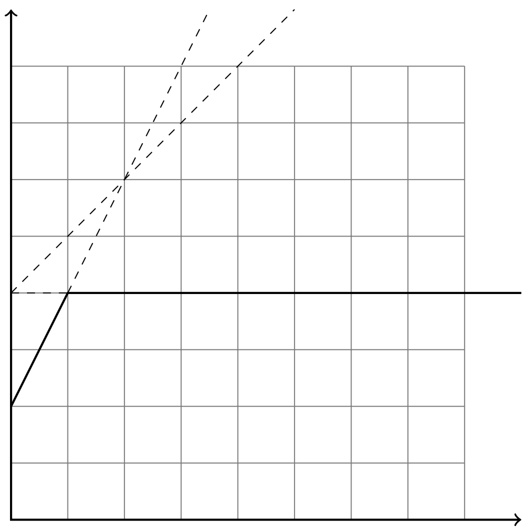
The factorization is

$$1x^2 \oplus 3x \oplus 5 = 1(x \oplus 2)^2.$$

The graphs of the three terms of $f(x)$ intersect in a single point.

The factorization of $f(x)$ contains a single linear factor twice, so $f(x)$ has a root of multiplicity 2. This is explained by the fact that the differences between consecutive coefficients of $f(x)$ are both 2.

10. Graph $f(x) = 2x^2 \oplus 4x \oplus 4$. Find a factorization in the form $f(x) = a(x \oplus r)(x \oplus s)$, or show that one does not exist.



We can factor out a 2 to obtain

$$f(x) = 2(x^2 \oplus 2x \oplus 2).$$

However, $x^2 \oplus 2x \oplus 2$ does not factor. There are no r and s with minimum 2 and sum 2.

11. Can you find a tropical polynomial which has the same graph as $f(x) = 2x^2 \oplus 4x \oplus 4$, but which can be factored?

The polynomial $2x^2 \oplus 3x \oplus 4 = 2(x \oplus 1)^2$ has the same graph as $f(x)$.

The ***Tropical Fundamental Theorem of Algebra*** *says that, for every tropical polynomial $f(x)$, there is a unique tropical polynomial $\bar{f}(x)$ with the same graph (and therefore determining the same function) which can be factored into linear factors. We sometimes say "the roots of $f(x)$" when we really mean "the roots of $\bar{f}(x)$."*

12. If $f(x) = ax^2 \oplus bx \oplus c$, then $\bar{f}(x) = ax^2 \oplus Bx \oplus c$ for some B. Find a formula for B in terms of a, b, and c. There are two different cases to consider.

In order to be able to factor

$$f(x) = a(x^2 \oplus (b-a)x \oplus (c-a)),$$

we need to find r and s such that $\min(r, s) = b - a$ and $r + s = c - a$. This is possible if and only if $2(b-a) \leq c - a$ or, equivalently, if $b \leq (a+c)/2$.

Case 1: If $b \leq (a+c)/2$, then $\bar{f}(x) = f(x)$ and $B = b$.

Case 2: If $b > (a+c)/2$, then

$$\begin{aligned}\bar{f}(x) &= ax^2 \oplus \left(\frac{a+c}{2}\right) x \oplus c \\ &= a\left(x \oplus \frac{c-a}{2}\right)^2\end{aligned}$$

has the same graph as $f(x)$, so $B = (a+c)/2$.

We can summarize both cases by saying that $B = \min(b, (a+c)/2)$.

13. State a tropical quadratic formula in terms of a, b, c for the roots x of a tropical polynomial $f(x) = ax^2 \oplus bx \oplus c$ (that is, the roots of the corresponding $\bar{f}$). There are once again two separate cases.

Case 1: If $b \leq (a+c)/2$, then $\bar{f}(x) = f(x)$ has roots $b-a$ and $c-b$, so that

$$\bar{f}(x) = a(x \oplus (b-a))(x \oplus (c-b)).$$

Case 2: If $b > (a+c)/2$, then $\bar{f}(x)$ has root $(c-a)/2$, with multiplicity 2, so that

$$\bar{f}(x) = a\left(x \oplus \frac{c-a}{2}\right)^2.$$

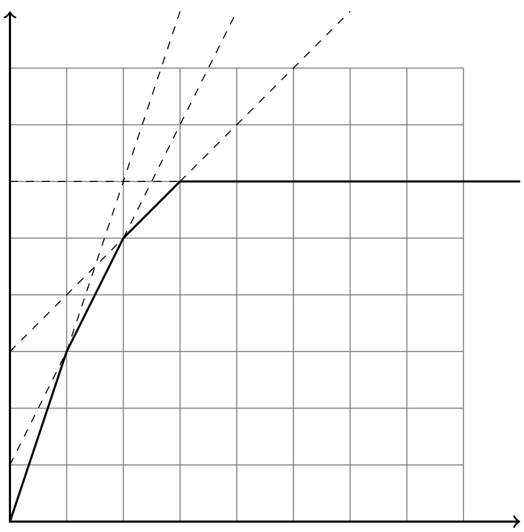
It is interesting to note that the condition $2b < a+c$ for there to be two distinct roots, when written in tropical notation, becomes $b^2 < ac$, which is reminiscent of the similar discriminant condition for standard polynomials.

2.2 Tropical cubic polynomials

14. For each cubic polynomial below,

- sketch the graph of the polynomial,
- use the graph to find the roots of the polynomial, and
- write (and expand) a product of linear factors with the same graph as the given polynomial.

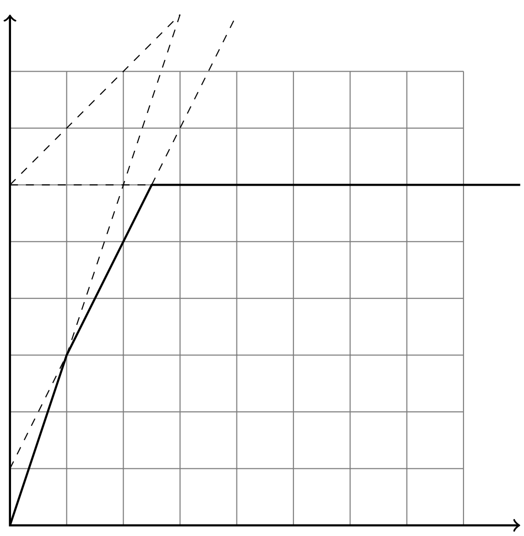
a) $f(x) = x^3 \oplus 1x^2 \oplus 3x \oplus 6$



The roots are 1, 2, and 3, yielding a factorization

$$\begin{aligned}\bar{f}(x) &= (x \oplus 1)(x \oplus 2)(x \oplus 3) \\ &= x^3 \oplus 1x^2 \oplus 3x \oplus 6.\end{aligned}$$

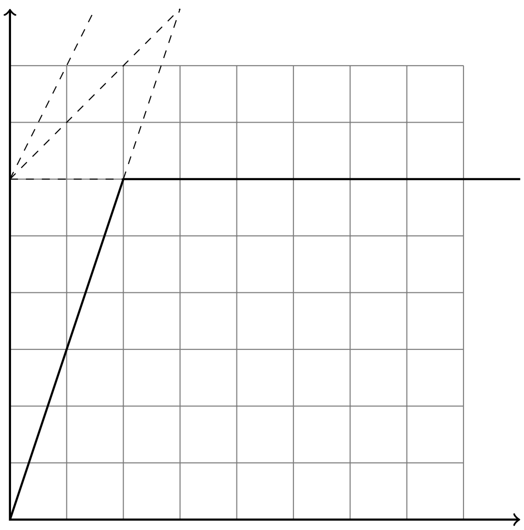
b) $g(x) = x^3 \oplus 1x^2 \oplus 6x \oplus 6$



The roots are 1, 2.5, and 2.5, yielding a factorization

$$\begin{aligned}\bar{f}(x) &= (x \oplus 1)(x \oplus 2.5)^2 \\ &= x^3 \oplus 1x^2 \oplus 3.5x \oplus 6.\end{aligned}$$

c) $h(x) = x^3 \oplus 6x^2 \oplus 6x \oplus 6$



The roots are 2, 2, and 2, yielding a factorization

$$\begin{aligned}\bar{f}(x) &= (x \oplus 2)^3 \\ &= x^3 \oplus 2x^2 \oplus 4x \oplus 6.\end{aligned}$$

15. If $f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d$, then $\bar{f}(x) = ax^3 \oplus Bx^2 \oplus Cx \oplus d$ for some B and C. With the preceding examples as a guide, find formulas for B and C in terms of a, b, c, and d.

$$B = \min\left(b, \frac{a+c}{2}, \frac{2a+d}{3}\right)$$

$$C = \min\left(c, \frac{b+d}{2}, \frac{a+2d}{3}\right)$$

2.3 General tropical polynomials

16. Can you guess the roots of the following polynomial?

$$f(x) = 3x^6 \oplus 4x^5 \oplus 2x^4 \oplus x^3 \oplus 1x^2 \oplus 4x \oplus 5$$

We have

$$\bar{f}(x) = 3x^6 \oplus 2x^5 \oplus 1x^4 \oplus x^3 \oplus 1x^2 \oplus 3x \oplus 5,$$

so the roots are $-1, -1, -1, 1, 2, 2$.

17. If

$$f(x) = a_n x^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0,$$

then

$$\bar{f}(x) = a_n x^n \oplus A_{n-1}x^{n-1} \oplus \cdots \oplus A_2 x^2 \oplus A_1 x \oplus a_0.$$

Can you find a formula for each A_j in terms of the a_i?

$$A_j = \min_{l \le j < k} \left(\frac{a_l - a_k}{k - l}(k - j) + a_k \right)$$
$$= \min_{l \le j < k} \left(a_l \left(\frac{k - j}{k - l} \right) + a_k \left(\frac{j - l}{k - l} \right) \right),$$

an appropriately weighted average of some a_l and a_k, with $l \le j < k$.

How about formulas for the roots $r_1, r_2, \ldots, r_n$?

The roots are simply the differences between consecutive coefficients of $\bar{f}(x)$. That is,

$$r_i = A_i - A_{i-1}$$

(where we set $A_n = a_n$ and $A_0 = a_0$).

Can you find a geometric interpretation of these formulas in terms of the points $(-i, a_i)$, for $0 \le i \le n$?

The inequality

$$A_j \le \frac{a_l - a_k}{k - l}(k - j) + a_k$$

(for $l \le j < k$) states that the point $(-j, A_j)$ must lie on or below the line segment between the points $(-k, a_k)$ and $(-l, a_l)$. This makes it easy to find the A_j using a graph of the points $(-i, a_i)$ for $0 \le i \le n$.