Introduction to Tropical Mathematics

Los Angeles Math Circle, November 7, 2010
Bryant Mathews, Azusa Pacific University

1 Preliminaries

In tropical arithmetic, we define new addition and multiplication operations on the real numbers. The **tropical sum** of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

while the **tropical product** of two numbers is their sum:

$$x \odot y = x + y.$$

1. Which of the following properties hold in tropical arithmetic? Prove or find a counterexample. ("Addition" and "multiplication" below mean *tropical* addition and *tropical* multiplication.)

- Addition is associative: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
- Addition is commutative: $x \oplus y = y \oplus x$.
- There exists an additive identity: There exists a number n such that $x \oplus n = x$ for all numbers x.

- True. $\min(\min(x, y), z) = \min(x, y, z) = \min(x, \min(y, z))$
- True. $\min(x, y) = \min(y, x)$
- False. Suppose such an n exists. Then $(n + 1) \oplus n = n \neq (n + 1)$.

2. What property (in terms of standard arithmetic) would an additive identity n have to satisfy? If a number had that property, what would you want to call that "number"? Let's expand our number set to include the real numbers *and* this new number. How will you define the tropical sum and tropical product of this new number with a real number or with itself?

We need $\min(x, n) = x$ for all real numbers x. This means that n must be greater than or equal to every real number x. We can think of infinity (∞) as being such a number. We define

$$\infty \oplus x = \min(\infty, x) = x$$
$$\infty \oplus \infty = \min(\infty, \infty) = \infty$$
$$\infty \odot x = \infty + x = \infty$$
$$\infty \odot \infty = \infty + \infty = \infty$$

3. Which of the following properties hold in tropical arithmetic? Prove or find a counterexample.

- Additive inverses exist: For each number x, there exists a number y such that $x \oplus y = n$, where n is the additive identity.
- Multiplication is associative: $(x \odot y) \odot z = x \odot (y \odot z)$.
- Multiplication is commutative: $x \odot y = y \odot x$.
- There exists a multiplicative identity: There exists a number i such that $x \odot i = x$ for all numbers x.
- Multiplicative inverses exist: For each number x not equal to the additive identity, there exists a number y such that $x \odot y = i$, where i is the multiplicative identity.
- Multiplication distributes over addition: $x \odot (y \oplus z) = x \odot y \oplus x \odot z$.

- False. Unless $x = \infty$, there is no y such that $x \oplus y = \infty$, i.e., such that $\min(x, y) = \infty$.
- True. $(x + y) + z = x + (y + z)$
- True. $x + y = y + x$
- True. The multiplicative identity is 0: $x \odot 0 = x + 0 = x$.
- True. For $x \neq \infty$, $x \odot (-x) = x + (-x) = 0$.
- True. $x + \min(y, z) = \min(x + y, x + z)$

4. Complete the tropical addition and multiplication tables below. In the blank spaces in the header rows and columns, write the appropriate (additive or multiplicative) identities:

$\oplus$	1	2	3	4	∞
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
∞	1	2	3	4	∞

$\odot$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

5. Expand $(x \oplus y)^2$ and $(x \oplus y)^3$. Can you simplify the resulting expressions? (Are all of the terms in the expansions necessary?)

We expand, representing tropical multiplication by juxtaposition:

$$(x \oplus y)^2 = x^2 \oplus xy \oplus yx \oplus y^2 = x^2 \oplus xy \oplus y^2 = x^2 \oplus y^2$$
$$(x \oplus y)^3 = x^3 \oplus x^2y \oplus xy^2 \oplus y^3 = x^3 \oplus y^3$$

These are called the "freshman's dream" formulas. The final equals sign in the first line is justified by the inequalities $x + y \geq 2x$ or $x + y \geq 2y$. Similarly for the final equals sign in the second line.

A shorter proof of $(x \oplus y)^n = x^n \oplus y^n$ proceeds as follows:

$$n \cdot \min(x, y) = \min(n \cdot x, n \cdot y).$$

2 Tropical Polynomials

It follows from the **Fundamental Theorem of Algebra** that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with **real coefficients**. For example,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x^2 + 2x + 5)(x - 2)(x + 4)^2.$$

Over the complex numbers, any such polynomial can be factored completely into polynomials of degree 1 with **complex coefficients**. For the example above,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x + 1 - 2i)(x + 1 + 2i)(x - 2)(x + 4)^2.$$

The factors can be determined by computing the **roots** (or the "zeros") of the polynomial. The polynomial above has roots

$$-1 + 2i, -1 - 2i, 2, -4, -4.$$

We say that the root -4 has *multiplicity* 2. There is a quadratic formula for determining the roots of a polynomial of degree 2, as well as cubic and quartic formulas for degrees 3 and 4. However, starting with degree 5, there is no longer a nice formula which enables us to find the roots of every polynomial. For polynomials of large degree, we generally must settle for approximate roots, found by a computer.

A tropical polynomial is an expression formed by (tropically) adding and/or multiplying tropical numbers (i.e. real numbers or ∞) and a variable x. A tropical polynomial can be written in the form

$$(a_n \odot x^n) \oplus \cdots \oplus (a_2 \odot x^2) \oplus (a_1 \odot x) \oplus (a_0).$$

For convenience, we represent tropical multiplication by juxtaposition, in the usual manner:

$$a_n x^n \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0.$$

Does the Fundamental Theorem of Algebra (or something close to it) hold for tropical polynomials? Is there a tropical quadratic formula for finding the roots of quadratic polynomials? What about a cubic formula? For polynomials of large degree, must we rely on a computer to find roots and factor, or can we do it ourselves?!

2.1 Tropical quadratic functions

6. Draw a precise graph of the tropical polynomial function $f(x) = x^2 \oplus 1x \oplus 4$. You may find it helpful to first rewrite the tropical polynomial (into an expression involving standard operations) using the definitions of $\oplus$ and $\odot$. Now, try to factor the tropical polynomial $x^2 \oplus 1x \oplus 4$ into linear (degree 1) factors, i.e.

$$x^2 \oplus 1x \oplus 4 = (x \oplus r)(x \oplus s).$$

The numbers r and s are called the **roots** of the tropical polynomial. Note that we use $\oplus$ because we do not have a tropical subtraction. Do you notice any relationship between the graph and the factorization? Can you see the roots in the graph?

In standard terms, $f(x) = \min(2x, 1 + x, 4)$. See graph paper for the graph.

Note that

$$(x \oplus r)(x \oplus s) = x^2 \oplus (r \oplus s)x \oplus (rs).$$

In order for this to equal $x^2 \oplus 1x \oplus 4$, we must have $\min(r, s) = 1$ and $r + s = 4$. We take $r = 1$ and $s = 3$:

$$f(x) = x^2 \oplus 1x \oplus 4 = (x \oplus 1)(x \oplus 3).$$

The roots are the x-values of the corners of the graph.

7. Repeat problem 6 for the tropical polynomial function $f(x) = -2x^2 \oplus x \oplus 8$, but with factorization of the form $a(x \oplus r)(x \oplus s)$. Can you see the roots r, s in the graph? How are the roots related to the coefficients of $f(x)$?

We first factor out -2: $f(x) = -2(x^2 \oplus 2x \oplus 10)$. Proceeding as in 6, we obtain

$$f(x) = -2(x \oplus 2)(x \oplus 8).$$

Note that the roots are the differences between consecutive coefficients: $2 = 0 - (-2)$, $8 = 8 - 0$.

8. Can you find a tropical polynomial function $f(x)$ with a value of 7 for all sufficiently large x and with roots of 4 and 5?

We are looking for $f(x) = ax^2 \oplus bx \oplus c$. We need $f(\infty) = 7$, so the constant term $c = 7$. We subtract 5 to get $b = 2$ and then 4 to get $a = -2$:

$$f(x) = -2x^2 \oplus 2x \oplus 7.$$

Note that it does not work to subtract the roots in the other order: indeed,

$$-2x^2 \oplus 3x \oplus 7 \neq -2(x \oplus 4)(x \oplus 5).$$

The polynomial $-2x^2 \oplus 3x \oplus 7$ does not factor algebraically, although it defines the same function as $-2(x \oplus 4.5)^2 = -2x^2 \oplus 2.5x \oplus 7$.

9. Repeat problem 6 for the tropical polynomial function $f(x) = 1x^2 \oplus 3x \oplus 5$. How is this graph

different from the others? How is this factorization different from the others? How are the roots related to the coefficients of $f(x)$?

All three lines involved in the graph intersect at the same point. The factorization contains the same factor twice

$$f(x) = 1x^2 \oplus 3x \oplus 5 = 1(x \oplus 2)^2.$$

This polynomial has a double root at $x = 2$.

10. Repeat problem 6 for the tropical polynomial function $f(x) = 2x^2 \oplus 4x \oplus 4$. What happens?

We can factor out a 2: $f(x) = 2(x^2 \oplus 2x \oplus 2)$. However, the polynomial does not factor into linear factors. There are no numbers r, s which have a minimum of 2 and a sum of 2.

11. Can you find a tropical polynomial which determines the same graph as $f(x) = 2x^2 \oplus 4x \oplus 4$, but which can be factored?

The graph of $f(x)$ looks like it has a double root at $x = 1$, so we guess that $g(x) = 2(x \oplus 1)^2$ has the same graph as $f(x)$ (and defines the same function). Expanding, we get $g(x) = 2x^2 \oplus 3x \oplus 4$.

*The **Tropical Fundamental Theorem of Algebra** says that for every tropical polynomial $f(x)$, there is a unique tropical polynomial $\bar{f}(x)$ with the same graph (or determining the same function) which can be factored into linear factors. We sometimes say "the roots of $f(x)$" to mean "the roots of $\bar{f}(x)$".*

12. If $f(x) = ax^2 \oplus bx \oplus c$, can you find a formula for $\bar{f}(x)$? There are two different cases. (Hint: When does $f(x) = \bar{f}(x)$?)

In order to factor $f(x) = a(x^2 \oplus (b-a)x \oplus (c-a))$, we need to find r, s such that $\min(r, s) = b-a$ and $r + s = c - a$. This can be done if and only if $b - a \leq \frac{c-a}{2}$, or $b \leq \frac{c+a}{2}$.

Case 1: $b \leq \frac{c+a}{2}$. In this case, $f = \bar{f}$.

Case 2: $b > \frac{c+a}{2}$. In this case, the linear term bx does not contribute to the value of $f(x)$. We see from the graph that $\bar{f}(x)$ has a double root at the x for which $a + 2x = c$, or $x = (c - a)/2$. Thus $\bar{f}(x) = a\left(x \oplus \frac{c-a}{2}\right)^2$.

13. State a tropical quadratic formula in terms of a, b, c for the roots x of a polynomial function $f(x) = ax^2 \oplus bx \oplus c$. There are once again two separate cases.

Case 1: $b \leq \frac{c+a}{2}$. Reasoning as above, we have $f(x) = \bar{f}(x)$ with roots $x = (c-a)-(b-a) = c-b$ and $x = b - a$.

Case 2: $b > \frac{c+a}{2}$. As shown above, $x = \frac{c-a}{2}$ is a double root.

2.2 Tropical cubic functions

15. Consider the cubic polynomial functions

$$\begin{aligned} f(x) &= x^3 \oplus \ \ x^2 \oplus 1x \oplus 3 \\ g(x) &= x^3 \oplus 1x^2 \oplus 1x \oplus 2 \\ h(x) &= x^3 \oplus 3x^2 \oplus 3x \oplus 3. \end{aligned}$$

For each function,

- Sketch the graph.
- Use the graph to find the roots.
- Write a product of linear factors which has the same graph as the given polynomial. Expand the product to obtain $\bar{f}(x), \bar{g}(x), \bar{h}(x)$. Which of the original polynomials are themselves factorable?

Using the graphs, we find the following roots:

$$\begin{aligned} \bar{f}(x) &= (x \oplus 0)(x \oplus 1)(x \oplus 2) &&= x^3 \oplus \ \ \ \ x^2 \oplus 1x \oplus 3 \\ \bar{g}(x) &= (x \oplus 0.5)^2(x \oplus 1) &&= x^3 \oplus 0.5x^2 \oplus 1x \oplus 2 \\ \bar{h}(x) &= (x \oplus 1)^3 &&= x^3 \oplus \ \ 1x^2 \oplus 2x \oplus 3 \end{aligned}$$

16. Consider a general tropical cubic function $f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d$. With the preceding examples as a guide, can you find a method for obtaining the factorable polynomial $\bar{f}(x)$ (and thereby obtaining the roots of f) directly from the coefficients a, b, c, d, without drawing the graph of $f(x)$? (If you need a hint, you may find it helpful to draw the points $(0, a), (1, b), (2, c), (3, d)$ on a coordinate plane and to consider line segments between these points.)

In order for a polynomial $f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d$ to be factorable, no point among $(0, a), (1, b), (2, c), (3, d)$ may lie below a line segment connecting two others. We lower these points as necessary until they lie on a sort of convex hull of the original points. The resulting polynomial is $\bar{f}(x)$.

17. Consider $f(x) = 6x^7 \oplus 8x^6 \oplus 2x^5 \oplus 2x^4 \oplus 6x^3 \oplus 6x^2 \oplus 7x \oplus 6$. Test your method from 16 by finding $\bar{f}(x)$ (without drawing the graph of $f(x)$). What are the roots of $f(x)$?

Applying the observation from 16, we find that

$$\begin{aligned} \bar{f}(x) &= 6x^7 \oplus 4x^6 \oplus 2x^5 \oplus 2x^4 \oplus 3x^3 \oplus 4x^2 \oplus 5x \oplus 6 \\ &= (x \oplus -2)^2(x \oplus 0)(x \oplus 1)^4, \end{aligned}$$

so that f has roots of $-2, 0, 1$, with respective multiplicities of $2, 1, 4$.