

# Warm-Up: Cosa Nostra

Prepared by Mark on March 31, 2026

## Instructor's Handout

This handout contains solutions and notes.

Recompile without solutions before distributing.

### Problem 1:

There are 36 gangsters in a certain district of Chicago. Some pairs of gangsters have feuds.

- Each gangster is part of at least one outfit, and no two outfits share the same members.
- If two gangsters are both in one outfit, there is no feud between them.
- A gangster that is not in a certain outfit must have a feud with at least one of its members.

What is the maximum number of outfits that can exist in this district?

### Solution:

**Definition:** Let the *authority* of a gangster be the number of outfits they are a part of.

**Lemma:** Say two gangsters have a feud. Label them  $x$  and  $y$  so that  $\text{authority}(x) > \text{authority}(y)$ . Then, replacing  $y$  with a clone of  $x$  will strictly increase the number of outfits.

If  $\text{authority}(x) = \text{authority}(y)$ , replacing  $y$  with  $x$  will not change the number of outfits.

**Proof:** Let  $A$  be the set of outfits that  $y$  is a part of, and  $B$  its complement (that is, all outfits that  $a$  is *not* a part of). If we delete  $y$ ...

- all outfits in  $B$  remain outfits.
- some outfits in  $A$  cease to be outfits (as they are no longer maximal)

Also, no new outfits are formed. If a new outfit  $o$  contains any enemies of  $y$ , it existed previously and is a member of  $B$ . If  $o$  contains no enemies of  $y$ , it must have contained  $y$  prior to deletion, and is thus a member of  $A$ . Therefore, the number of outfits is reduced by at most  $\text{authority}(y)$  when  $y$  is deleted.

If we add a clone of  $x$  after deleting  $y$  (this clone has a feud with  $x$ ), All previous outfits remain outfits, and  $\text{authority}(x)$  new outfits are created. Therefore, replacing  $y$  with a clone of  $x$  strictly increases the number of outfits that exist. We thus conclude that in the maximal case, all pairs of feuding gangsters have equal authority.

**Solution:** Consider an arbitrary gangster  $g$ . By the previous lemma, we can replace all gangsters  $g$  has a feud with clones of itself. Repeat this for all gangsters, and we are left with groups of feuding gangsters who are friends with everyone outside their group. The total number of outfits is the product of the sizes of these groups.

This problem is now equivalent to the "Partition Products" warm-up. We want the list of numbers whose sum is 36 and whose product is maximal. The solution is to form 12 groups of three gangsters for a total of  $3^{12}$  outfits.