Combinatorics

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Instructor's Handout

This file contains solutions and notes.
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Part 1: Getting started

An **ordered** arrangement of objects is called a *permutation*.
An **unordered** selection of objects is called a *combination*.[1]
All the following problems involve permutations.

Problem 1:
How many different ways are there to rearrange the letters ABCDE?

Solution

$5 \times 4 \times 3 \times 2 \times 1 = 120$

Problem 2:
How many different ways are there to arrange the letters ABCDEFG...XYZ?
You don't need to fully evaluate your answer, it is a *very* big number.
Hint: Look at Problem 1 again, and try to create a general strategy.

Note for Instructors

A hint for students that are stuck:
In Problem 1, start with five blank spaces. How many choices are there for A's position?
Once A is placed, how many are left for B?

[1] A "combination lock" cares about the order of its digits, so its name is inaccurate. Such an object is actually a *permutation* lock!

Definition 3:
The *factorial* of a positive integer x is $x \times (x-1) \times ... \times 1$. We denote this $x!$.
For example, $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$.

Problem 4:
Compute $\frac{10!}{8!}$

Problem 5:
Convince yourself that $(n+1)! = n! \times (n+1)$, and use this fact to show that $0! = 1$.

Problem 6:
How many ways are there to choose three student council officers from a class of 20 students?
How many ways are there to choose a president, a vice-president, and a treasurer from the same class?
Hint: You answers should be different. In which case does order matter?

Note for Instructors

Have your students consider the non-unique arrangements and count how many are redundant.

Problem 7:
Say you have 4 red balls and 3 green balls. How many different ways can you arrange them on the table in font of you?

Solution

Consider the sequence RRRRGGG. There are 4! ways to rearrange the red balls, and 3! ways to rearrange the green balls. This is true for any sequence.
So, our solution is $\frac{7!}{3!4!}$.

Problem 8:
How many *unique* anagrams can we create from the word CRESCENDO?

Solution

CRESCENDO = CC EE RSNDO, our solution is $\frac{9!}{2!2!} = 90720$

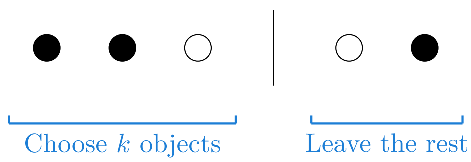
Problem 9:
Given the letters ABCDE, how many different three-letter words can we make without repeating letters?

Part 2: Permutations

It would be convenient to have a general tool for counting permutations. Let us try to create one. (Remember, permutations are *ordered* arrangements of objects.)
First, let's create a function ${}_nP_k$, which tells us how many k-object permutations we can choose from a group of n objects.

Problem 10:
What is ${}_5P_3$?
Hint: See Problem 9

"Choosing k items from n" is a lot like splitting our n objects into two groups: those we choose, and those we don't.



If we rearrange these, we get different permutations. How can we count them?

Problem 11:
Using the above diagram, create a formula for ${}_nP_k$.
Hint: We're counting *permutations*, so the order of items in the first group matters.

Solution

${}_nP_k = \frac{n!}{(n-k)!}$
There are $n!$ possible arrangements of n objects. However, since the order of the elements not chosen does not matter, we'll end up with $(n-k)!$ redundant orderings of each.

Part 3: Combinations

Now, let's count *combinations*.
Here, we only care about *which* items we choose—not the order in which we choose them. We'll make a function ${}_nC_k$ ("n choose k"), which will tell us how many different ways we can choose k items from a set of n.

Problem 12:
Find an expression for ${}_nC_k$ by modifying your definition of ${}_nP_k$.

Usually, ${}_nC_k$ is written as $\binom{n}{k}$. This is also called the *binomial coefficient*.

Problem 13:
Say you have a few coins on the table in font of you:

- 8 identical 1-kopek[2] coins
- 3 identical 2-kopek coins
- 6 identical 5-kopek coins
- 4 identical 10-kopek coins

How many distinct ways are there to arrange these coins in a row?

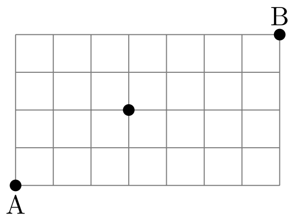
Problem 14:
Now, derive the *multinomial coefficient* $\binom{n}{k_1, k_2, \ldots, k_m}$.
The multinomial coefficient tells us how many distinct ways there to arrange n objects of m classes, where each class i contains k_i identical objects.
Hint: In Problem 13, $n = 21$ and $(k_1, k_2, k_3, k_4) = (8, 3, 6, 4)$.
So, the solution to Problem 13 should be given by the multinomial coefficient $\binom{21}{8,3,6,4}$.

[2] Russian currency. Comparable to a penny, since 100 kopeks make a ruble.

Part 4: Applications

Problem 15:
How many ways can a class of 27 people be seated in 30 seats?

Problem 16:
The following is the map of a city. Each line is a one-way road, you can only drive up or right.
How many different paths can you take from A to B?
How many of them go through the center point?



Problem 17:
How many ways can you put 19 identical balls into 6 bins, leaving no bin empty?

Problem 18:
Given an exam with 4 problems, how many ways are there to assign positive point values to each problem so that the exam contains a total of 100 points?

Problem 19:
How many ways can we split the number 2016 into a sum of positive integers?
Consider $2016 + 1$ and $1 + 2016$ distinct sums. Order matters.

Solution

Split 2016 into ones, and put a "bit" between each pair.
This gives us 2^{2015} positions to place a bar, and thus 2^{2016} possible sums.

You could also sum over the usual stars-and-bars technique to get the same result.
Showing that they're equal could be a good bonus problem!

Problem 20:
A staircase must be built up a wall. It will start 4.5 meters away from the wall, which is 1.5 meters tall. The height of each step is exactly 30 centimeters. The width of each step must be an integer multiple of 50 centimeters. In how many ways can the staircase be constructed?