Generating Functions

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Based on a handout by Aaron Anderson

Part 1: Introduction

Definition 1:
Say we have a sequence $a_0, a_1, a_2, ...$
The *generating function* of this sequence is defined as follows:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

Under some circumstances, this sum does not converge, and thus $A(x)$ is undefined.
However, we can still manipulate this infinite sum to get useful results even if $A(x)$ diverges.

Problem 2:
Let $A(x)$ be the generating function of the sequence a_n,
and let $B(x)$ be the generating function of the sequence b_n.
Find the sequences that correspond to the following generating functions:

- $cA(x)$

- $xA(x)$

- $A(x) + B(x)$

- $A(x)B(x)$

Problem 3:
Assuming $|x| < 1$, show that
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$
Hint: use some clever algebra. What is $x \times (1 + x + x^2 + \dots)$?

Problem 4:
Let $A(x)$ be the generating function of the sequence a_n.
Find the sequence that corresponds to the generating function $\frac{A(x)}{1-x}$

Problem 5:
Find short expressions for the generating functions for the following sequences:
- $1, 0, 1, 0, \dots$
- $1, 2, 4, 8, 16, \dots$
- $1, 2, 3, 4, 5, \dots$

Part 2: Fibonacci

Definition 6:

The *Fibonacci numbers* are defined by the following recurrence relation:
- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$

Problem 7:

Let $F(x)$ be the generating function that corresponds to the Fibonacci numbers.
Find the generating function of $0, f_0, f_1, \dots$ in terms of $F(x)$.
Call this $G(x)$.

Problem 8:

Find the generating function of $0, 0, f_0, f_1, \dots$ in terms of $F(x)$. Call this $H(x)$.

Problem 9:

Calculate $F(x) - G(x) - H(x)$ using the recurrence relation that we used to define the Fibonacci numbers.

Problem 10:
Using the problems on the previous page, find $F(x)$ in terms of x.

Definition 11:
A *rational function* f is a function that can be written as a quotient of polynomials. That is, $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials.

Problem 12:
Solve the equation from **??**fibo> for $F(x)$, expressing it as a rational function.

Definition 13:
Partial fraction decomposition is an algebreic technique that works as follows:
If $p(x)$ is a polynomial and a and b are constants, we can rewrite the rational function $\frac{p(x)}{(x-a)(x-b)}$ as follows:

$$\frac{p(x)}{(x-a)(x-b)} = \frac{c}{x-a} + \frac{d}{x-b}$$

where c and d are constants.

Problem 14:
Now that we have a rational function for $F(x)$,
find a closed-form expression for its coefficients using partial fraction decomposition.

Problem 15:
Using problems from the introduction and **??**, find an expression for the coefficients of $F(x)$ (and this, for the Fibonacci numbers).

Problem 16: Bonus

Repeat the method of recurrence, generating function, partial fraction decomposition, and geometric series to find a closed form for the following sequence:

$$a_0 = 1; \quad a_{n+1} = 2a_n + n$$

Hint: When doing partial fraction decomposition with a denominator of the form $(x - a)^2(x - b)$, you may need to express your expression as a sum of three fractions: $\frac{c}{(x-a)^2} + \frac{d}{x-a} + \frac{e}{x-b}$."

Part 3: Dice

Definition 17:
A *die* is a device that randomly selects a positive integer from a finite list of options. For example, the standard 6-sided die selects a value from $[1, 2, 3, 4, 5, 6]$. We may have many sides with the same value, as in $[1, 1, 2, 3]$.

To describe a die with a generating function, let a_k be the number of times k appears as a side of the die and consider $a_0 + a_1 x + x_2 x^2 + ...$

A die has a finite number of sides, so this will be a regular polynomial.

Problem 18:
What is the generating function of the standard 6-sided die?

Problem 19:
What is the generating function of the die with sides $[1, 2, 3, 5]$?

Problem 20:
Let $A(x)$ and $B(x)$ be the generating functions of two dice.
What is the significance of $A(1)$?

Problem 21:
Using formulas we found earlier, show that the kth coefficient of $A(x)B(x)$ is the number of ways to roll k as the sum of the two dice.

Problem 22:
Find a generating function for the sequence $c_0, c_1, ...$, where c_k is the probability that the sum of the two dice is k.

Problem 23:
Using generating functions, find two six-sided dice whose sum has the same distribution as the sum of two standard six-sided dice?
That is, for any integer k, the number if ways that the sum of the two nonstandard dice rolls as k is equal to the number of ways the sum of two standard dice rolls as k. *Hint:* factor polynomials.

Part 4: Coins

Consider the following problem:
"How many different ways can you make change for $0.50
using pennies, nickels, dimes, quarters and half-dollars?"

Most ways of solving this involve awkward brute-force approache that don't reveal anything
interesting about the problem: how can we change our answer if we want to make change for $0.51,
or $1.05, or some other quantity?

We can use generating functions to solve this problem in a general way.

Definition 24:
Let $p_0, p_1, p_2, ...$ be such that p_k is the number of ways to make change for k cents with only pennies.
Similarly, let...
- n_k be the number of ways to make change for k cents with only nickels;
- d_k be the number of ways using only dimes;
- q_k be the number of ways using only quarters;
- and h_k be the number of ways using only half-dollars.

Problem 25:
Let $p(x)$ be the generating function that corresponds to p_n.
Express $p(x)$ as a rational function.

Problem 26:
Modify **??** to find expressions for $n(x)$, $d(x)$, $q(x)$, and $h(x)$.

Definition 27:
Now, let $N(x)$ be the generating function for the sequence $n_0, n_1, ...$, where n_k is the number of ways to make change for k cents using pennies and nickels.
Similarly, let...
- let $D(x)$ be the generating function for the sequence using pennies, nickels, and dimes;
- let $Q(x)$ use pennies, nickels, dimes, and quarters;
- and let $H(x)$ use all coins.

Problem 28:
Express $N(x)$ as a rational function.

Problem 29:
Using the previous problem, write $D(x)$, then $Q(x)$, then $H(x)$ as rational functions.

Problem 30:
Using these generating functions, find recurrence relations for the sequences N_k, D_k, Q_k, and H_k.
Hint: Your recurrence relation for N_k should refer to the previous values of itself and some values of p_k. Your recurrence for D_k should refer to itself and N_k; the one for Q_k should refer to itself D_k; and the one for H_k should refer to itself and Q_k.

Problem 31:
Using these recurrence relations, fill following table and solve the original problem.

n	0	5	10	15	20	25	30	35	40	45	50
p_k											
N_k											
D_k											
Q_k											
H_k											