

De Bruijn Sequences

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Based on a handout by Glenn Sun

Part 1: Introduction

Example 1:

A certain electronic lock has two buttons: 0 and 1. It opens as soon as the correct two-digit code is entered, completely ignoring previous inputs. For example, if the correct code is 10, the lock will open once the sequence 010 is entered.

Naturally, there are $2^2 = 4$ possible combinations that open this lock.

If we don't know the lock's combination, we could try to guess it by trying all four combinations. This would require eight key presses: 0001101100.

Problem 2:

There is, of course, a better way.

Unlock this lock with only 5 keypresses.

Now, consider the same lock, now set with a three-digit binary code.

Problem 3:

How many codes are possible?

Problem 4:

Show that there is no solution with fewer than three keypresses

Problem 5:

What is the shortest sequence that is guaranteed to unlock the lock?

Hint: You'll need 10 digits.

Part 2: Words

Definition 6:

An *alphabet* is a set of symbols.

For example, $\{0, 1\}$ is an alphabet of two symbols, and $\{a, b, c\}$ is an alphabet of three.

Definition 7:

A *word* over an alphabet A is a sequence of symbols in that alphabet.

For example, 00110 is a word over the alphabet $\{0, 1\}$.

We'll let \emptyset denote the empty word, which is a valid word over any alphabet.

Definition 8:

Let v and w be words over the same alphabet.

We say v is a *subword* of w if v is contained in w .

In other words, v is a subword of w if we can construct v by removing a few characters from the start and end of w .

For example, 11 is a subword of 011, but 00 is not.

Definition 9:

Recall ???. Let's generalize this to the *n-subword problem*:

Given an alphabet A and a positive integer n , we want a word over A that contains all possible length- n subwords. The shortest word that solves a given n -subword problem is called the *optimal solution*.

Problem 10:

List all subwords of 110.

Hint: There are six.

Definition 11:

Let $\mathcal{S}_n(w)$ be the number of subwords of length n in a word w .

Problem 12:

Find the following:

- $\mathcal{S}_n(101001)$ for $n \in \{0, 1, \dots, 6\}$
- $\mathcal{S}_n(\text{abccac})$ for $n \in \{0, 1, \dots, 6\}$

Problem 13:

Let w be a word over an alphabet of size k .

Prove the following:

- $\mathcal{S}_n(w) \leq k^n$
- $\mathcal{S}_n(w) \geq \mathcal{S}_{n-1}(w) - 1$
- $\mathcal{S}_n(w) \leq k \times \mathcal{S}_{n-1}(w)$

Definition 14:

Let v and w be words over the same alphabet.

The word vw is the word formed by writing v after w .

For example, if $v = 1001$ and $w = 10$, vw is 100110 .

Problem 15:

Let F_k denote the word over the alphabet $\{0, 1\}$ obtained from the following relation:

$$F_0 = 0; \quad F_1 = 1; \quad F_k = F_{k-1}F_{k-2}$$

We'll call this the *Fibonacci word* of order k .

- What are F_3 , F_4 , and F_5 ?
- Compute S_0 through S_5 for F_5 .
- Show that the length of F_k is the $(k + 2)^{\text{th}}$ Fibonacci number.

Hint: Induction.

Problem 16:

Let C_k denote the word over the alphabet $\{0, 1\}$ obtained by concatenating the binary representations of the integers $0, \dots, 2^k - 1$. For example, $C_1 = 01$, $C_2 = 011011$, and $C_3 = 011011100101110111$.

- Compute \mathcal{S}_0 , \mathcal{S}_1 , \mathcal{S}_2 , and \mathcal{S}_3 for C_3 .
- Show that $\mathcal{S}_k(C_k) = 2^k - 1$.
- Show that $\mathcal{S}_n(C_k) = 2^n$ for $n < k$.

Hint: If v is a subword of w and w is a subword of u , v must be a subword of u . In other words, the “subword” relation is transitive.

Problem 17:

Convince yourself that C_{n+1} provides a solution to the n -subword problem over $\{0, 1\}$.

Note: C_{n+1} may or may not be an *optimal* solution—but it is a *valid* solution. Which part of ?? shows that this is true?

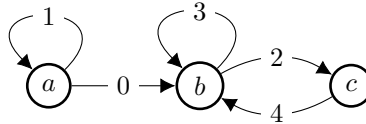
Part 3: De Bruijn Words

Before we continue, we'll need to review some basic graph theory.

Definition 18:

A *directed graph* consists of nodes and directed edges.

An example is shown below. It consists of three vertices (labeled a, b, c), and five edges (labeled $0, \dots, 4$).



Definition 19:

A *path* in a graph is a sequence of adjacent edges,

In a directed graph, edges a and b are adjacent if a ends at the node which b starts at.

For example, consider the graph above.

The edges 1 and 0 are adjacent, since you can take edge 0 after taking edge 1.

0 starts where 1 ends.

0 and 1, however, are not: 1 does not start at the edge at which 0 ends.

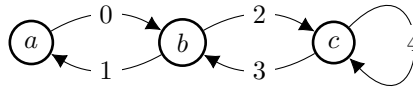
Definition 20:

An *Eulerian path* is a path that visits each edge of a graph exactly once.

An *Eulerian cycle* is an Eulerian path that starts and ends on the same node.

Problem 21:

Find the single unique Eulerian cycle in the graph below.



Theorem 22:

A directed graph contains an Eulerian cycle iff...

- There is a path between every pair of nodes, and
- every node has as many “in” edges as it has “out” edges.

If a graph contains an Eulerian cycle, it must contain an Eulerian path. (why?)

Some graphs contain an Eulerian path, but not a cycle. In this case, both conditions above must still hold, but the following exceptions are allowed:

- There may be at most one node where (number in – number out) = 1
- There may be at most one node where (number in – number out) = –1

Note: Either both exceptions occur, or neither occurs. Bonus problem: why?

We won't provide a proof of this theorem today. However, you should convince yourself that it is true: if any of these conditions are violated, why do we know that an Eulerian cycle (or path) cannot exist?

Definition 23:

Now, consider the n -subword problem over $\{0, 1\}$.

We'll call the optimal solution to this problem a *De Bruijn*¹ word of order n .

Problem 24:

Let w be the an order- n De Bruijn word, and denote its length with $|w|$.

Show that the following bounds always hold:

- $|w| \leq n2^n$
- $|w| \geq 2^n + n - 1$

Remark 25:

Now, we'd like to show that the length of a De Bruijn word is always $2^n + n - 1$

That is, that the optimal solution to the subword problem always has $2^n + n - 1$ letters.

We'll do this by construction: for a given n , we want to build a word with length $2^n + n - 1$ that solves the binary n -subword problem.

Definition 26:

Consider a n -length word w .

The *prefix* of w is the word formed by the first $n - 1$ letters of w .

The *suffix* of w is the word formed by the last $n - 1$ letters of w .

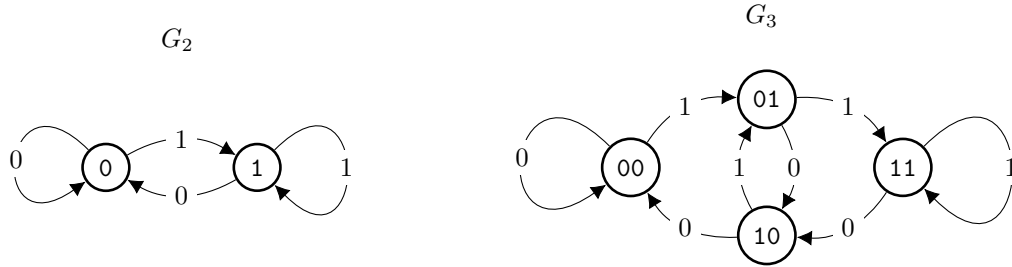
For example, the prefix of the word 1101 is 110, and its suffix is 101. The prefix and suffix of any one-letter word are both \emptyset .

Definition 27:

A *De Bruijn graph* of order n , denoted G_n , is constructed as follows:

- Nodes are created for each word of length $n - 1$.
- A directed edge is drawn from a to b if the suffix of a matches the prefix of b .
Note that a node may have an edge to itself.
- We label each edge with the last letter of b .

G_2 and G_3 are shown below.



¹Dutch. Rhymes with “De Grown.”

Problem 28:
Draw G_4 .

Problem 29:

- Show that G_n has 2^{n-1} nodes and 2^n edges;
- that each node has two outgoing edges;
- and that there are as many edges labeled 0 as are labeled 1.

Problem 30:

Show that G_4 always contains an Eulerian path.

Hint: ??

Theorem 31:

We can now easily construct De Bruijn words for a given n :

- Construct G_n ,
- find an Eulerian cycle in G_n ,
- then, construct a De Bruijn word by writing the label of our starting vertex, then appending the label of every edge we travel.

Problem 32:

Find De Bruijn words of orders 2, 3, and 4.

Let's quickly show that the process described in ?? indeed produces a valid De Bruijn word.

Problem 33:

How long will a word generated by the above process be?

Problem 34:

Show that a word generated by the process in ?? contains every possible length- n subword. In other words, show that $\mathcal{S}_n(w) = 2^n$ for a generated word w .

Remark 35:

- We found that ?? generates a word with length $2^n + n - 1$ in ??,
- and we showed that this word always solves the n -subword problem in ??.
- From ??, we know that any solution to the binary n -subword problem must have at least $2^n + n - 1$ letters.
- Finally, ?? guarantees that it is possible to generate such a word in any G_n .

Thus, we have shown that the process in ?? generates ideal solutions to the n -subword problem, and that such solutions always exist. We can now conclude that for any n , the binary n -subword problem may be solved with a word of length $2^n + n - 1$.

Part 4: Line Graphs

Problem 36:

Given a graph G , we can construct a graph called the *line graph* of G (denoted $\mathcal{L}(G)$) by doing the following:

- Creating a node in $\mathcal{L}(G)$ for each edge in G
- Drawing a directed edge between every pair of nodes a, b in $\mathcal{L}(G)$ if the corresponding edges in G are adjacent.

That is, if edge b in G starts at the node at which a ends.

Problem 37:

Draw the line graph for the graph below.

Have an instructor check your solution.



Definition 38:

We say a graph G is *connected* if there is a path between any two vertices of G .

Problem 39:

Show that if G is connected, $\mathcal{L}(G)$ is connected.

Definition 40:

Consider $\mathcal{L}(G_n)$, where G_n is the n^{th} order De Bruijn graph.

We'll need to label the vertices of $\mathcal{L}(G_n)$. To do this, do the following:

- Let a and b be nodes in G_n
- Let x be the first letter of a
- Let y , the last letter of b
- Let \bar{p} be the prefix/suffix that a and b share.

Note that $a = x\bar{p}$ and $b = \bar{p}y$,

Now, relabel the edge from a to b as $x\bar{p}y$.

Use these new labels to name nodes in $\mathcal{L}(G_n)$.

Problem 41:

Construct $\mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$. What do you notice?

Hint: What are $\mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$? We've seen them before!

You may need to re-label a few edges.

Part 5: Sturmian Words

A De Bruijn word is the shortest word that contains all subwords of a given length.

Let's now solve a similar problem: given an alphabet, we want to construct a word that contains exactly m distinct subwords of length n .

In general, this is a difficult problem. We'll restrict ourselves to a special case:

We'd like to find a word that contains exactly $m + 1$ distinct subwords of length m for all $m < n$.

Definition 42:

We say a word w is a *Sturmian word* of order n if $\mathcal{S}_m(w) = m + 1$ for all $m \leq n$.

We say w is a *minimal* Sturmian word if there is no shorter Sturmian word of that order.

Problem 43:

Show that the length of a Sturmian word of order n is at least $2n$.

Problem 44:

Construct R_3 by removing four edges from G_3 .

Show that each of the following is possible:

- R_3 does not contain an Eulerian path.
- R_3 contains an Eulerian path, and this path constructs a word w with $\mathcal{S}_3(w) = 4$ and $\mathcal{S}_2(w) = 4$.
- R_3 contains an Eulerian path, and this path constructs a word w that is a minimal Sturmian word of order 3.

Problem 45:

Construct R_2 by removing one edge from G_2 , then construct $\mathcal{L}(R_2)$.

- If this line graph has four edges, set $R_3 = \mathcal{L}(R_2)$.
- If not, remove one edge from $\mathcal{L}(R_2)$ so that an Eulerian path still exists and set R_3 to the resulting graph.

Label each edge in R_3 with the last letter of its target node.

Let w be the word generated by an Eulerian path in this graph, as before.

Attempt the above construction a few times. Is w a minimal Sturmian word?

Theorem 46:

We can construct a minimal Sturmian word of order $n \geq 3$ as follows:

- Start with G_2 , create R_2 by removing one edge.
- Construct $\mathcal{L}(G_2)$, remove an edge if necessary.
The resulting graph must have an 4 edges and an Eulerian path. Call this R_3 .
- Repeat the previous step to construct a sequence of graphs R_n .
 R_{n-1} is used to create R_n , which has $n + 1$ edges and an Eulerian path.
Label edges with the last letter of their target vertex.
- Construct a word w using the Eulerian path, as before.
This is a minimal Sturmian word.

For now, assume this theorem holds. We'll prove it in the next few problems.

Problem 47:

Construct a minimal Sturmian word of order 4.

Problem 48:

Construct a minimal Sturmian word of order 5.

Problem 49:

Argue that the words we get by ?? are minimal Sturmian words.

That is, the word w has length $2n$ and $\mathcal{S}_m(w) = m + 1$ for all $m \leq n$.