Warm-Up: Partition Products

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> **Instructor's Handout**
> This handout contains solutions and notes.
> Recompile without solutions before distributing.

Problem 1:

Take any positive integer n.

Now, write it as sum of smaller positive integers: $n = a_1 + a_2 + ...a_k$

Maximize the product $a_1 \times a_2 \times ... \times a_k$

Solution:

Interesting Solution:

Of course, all a_i should be greater than 1.

Also, all a_i should be smaller than four, since $x \leq x(x-2)$ if $x \geq 4$.

Thus, we're left with sequences that only contain 2 and 3.

Note that two twos are the same as one four, but we exclude fours for simplicity.

Finally, we see that $3^2 > 2^3$, so any three twos are better repackaged as two threes.

The best sequence a_i thus consists of a maximal number of threes followed by 0, 1, or 2 twos.

Calculus Solution:

First, solve this problem for equal, real a_i:

We know $n = \prod(a_i)$, thus $\ln(n) = \sum(\ln(a_i))$.

If all a_i are equal, we get $\ln(n) = k \times \ln\left(\frac{n}{k}\right)$.

Derive wrt k and set to zero to get $\ln\left(\frac{n}{k}\right) = 1$

So $k = \frac{n}{e}$ and $\frac{n}{k} = e \approx 2.7$

If we try to approximate this with integers, we get the same solution as above.