

Generating Functions

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Based on a handout by Aaron Anderson

Instructor's Handout

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Part 1: Introduction

Definition 1:

Say we have a sequence a_0, a_1, a_2, \dots

The *generating function* of this sequence is defined as follows:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Under some circumstances, this sum does not converge, and thus $A(x)$ is undefined.

However, we can still manipulate this infinite sum to get useful results even if $A(x)$ diverges.

Problem 2:

Let $A(x)$ be the generating function of the sequence a_n ,

and let $B(x)$ be the generating function of the sequence b_n .

Find the sequences that correspond to the following generating functions:

- $cA(x)$
- $xA(x)$
- $A(x) + B(x)$
- $A(x)B(x)$

Solution

- $cA(x)$ corresponds to ca_n
- $xA(x)$ corresponds to $0, a_0, a_1, \dots$
- $A(x) + B(x)$ corresponds to $a_n + b_n$
- $A(x)B(x)$ is $a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots$
Which corresponds to $c_n = \sum_{k=0}^n a_k b_{n-k}$

Problem 3:

Assuming $|x| < 1$, show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Hint: use some clever algebra. What is $x \times (1 + x + x^2 + \dots)$?

Solution

Let $S = 1 + x + x^2 + \dots$
 Then, $xS = x + x^2 + x^3 + \dots$
 So, $xS = S - 1$
 and $1 = S - xS = S(1 - x)$
 and $S = \frac{1}{1-x}$.

Problem 4:

Let $A(x)$ be the generating function of the sequence a_n .

Find the sequence that corresponds to the generating function $\frac{A(x)}{1-x}$

Solution

$$\begin{aligned} \frac{A(x)}{1-x} &= A(x)(1 + x + x^2 + \dots) \\ &= (a_0 + a_1x + a_2x^2 + \dots)(1 + x + x^2 + \dots) \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots \end{aligned}$$

Which corresponds to the sequence $c_n = \sum_{k=0}^n a_k$

Problem 5:

Find short expressions for the generating functions for the following sequences:

- 1, 0, 1, 0, ...
- 1, 2, 4, 8, 16, ...
- 1, 2, 3, 4, 5, ...

Solution

- 1, 0, 1, 0, ... corresponds to $1 + x^2 + x^4 + \dots$
 By Problem 3, this is $\frac{1}{1-x^2}$.
- 1, 2, 4, 8, 16, ... corresponds to $1 + 2x + (2x)^2 + \dots$
 By Problem 3, this is $\frac{1}{1-2x}$.
- 1, 2, 3, 4, 5, ... corresponds to $1 + 2x + 3x^2 + 4x^3 + \dots$
 This is equal to $(1 + x + x^2 + \dots)^2$, and thus is $\left(\frac{1}{1-x}\right)^2$

Part 2: Fibonacci

Definition 6:

The *Fibonacci numbers* are defined by the following recurrence relation:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$

Problem 7:

Let $F(x)$ be the generating function that corresponds to the Fibonacci numbers.

Find the generating function of $0, f_0, f_1, \dots$ in terms of $F(x)$.

Call this $G(x)$.

Solution

$$G(x) = xF(x)$$

Problem 8:

Find the generating function of $0, 0, f_0, f_1, \dots$ in terms of $F(x)$. Call this $H(x)$.

Solution

$$H(x) = x^2F(x)$$

Problem 9:

Calculate $F(x) - G(x) - H(x)$ using the recurrence relation that we used to define the Fibonacci numbers.

Solution

$$\begin{aligned} F(x) - G(x) - H(x) &= f_0 + (f_1 - f_0)x + (f_2 - f_1 - f_0)x^2 + (f_3 - f_2 - f_1)x^3 + \dots \\ &= f_0 + (f_1 - f_0)x \\ &= x \end{aligned}$$

Problem 10:

Using the problems on the previous page, find $F(x)$ in terms of x .

Solution

$$\begin{aligned} x &= F(x) - G(x) - H(x) \\ &= F(x) - xF(x) - x^2F(x) \\ &= F(x)(1 - x - x^2) \end{aligned}$$

So,

$$F(x) = \frac{x}{1 - x - x^2}$$

Definition 11:

A *rational function* f is a function that can be written as a quotient of polynomials.

That is, $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials.

Problem 12:

Solve the equation from ??fib> for $F(x)$, expressing it as a rational function.

Solution

$$\begin{aligned} F(x) &= \frac{-x}{x^2 + x - 1} = \frac{-x}{(x - a)(x - b)} \\ &= \frac{1 - \sqrt{5}}{2\sqrt{5}} \frac{1}{x - a} + \frac{-1 - \sqrt{5}}{2\sqrt{5}} \frac{1}{x - b} \end{aligned}$$

where

$$a = \frac{-1 + \sqrt{5}}{2}; \quad b = \frac{-1 - \sqrt{5}}{2}$$

Definition 13:

Partial fraction decomposition is an algebraic technique that works as follows:

If $p(x)$ is a polynomial and a and b are constants, we can rewrite the rational function $\frac{p(x)}{(x-a)(x-b)}$ as follows:

$$\frac{p(x)}{(x-a)(x-b)} = \frac{c}{x-a} + \frac{d}{x-b}$$

where c and d are constants.

Problem 14:

Now that we have a rational function for $F(x)$,

find a closed-form expression for its coefficients using partial fraction decomposition.

Solution

$$\begin{aligned} F(x) &= \left(\frac{1-\sqrt{5}}{2\sqrt{5}} \right) \left(\frac{-1}{a} \right) \left(\frac{1}{1-\frac{x}{a}} \right) + \left(\frac{-1-\sqrt{5}}{2\sqrt{5}} \right) \left(\frac{-1}{b} \right) \left(\frac{1}{1-\frac{x}{b}} \right) \\ &= \left(\frac{1}{\sqrt{5}} \right) \left(\frac{1}{1-\frac{x}{a}} \right) + \left(\frac{-1}{\sqrt{5}} \right) \left(\frac{1}{1-\frac{x}{b}} \right) \\ &= \frac{1}{\sqrt{5}} \left(1 + \frac{x}{a} + \left(\frac{x}{a} \right)^2 + \dots \right) - \frac{1}{\sqrt{5}} \left(1 + \frac{x}{b} + \left(\frac{x}{b} \right)^2 + \dots \right) \end{aligned}$$

Problem 15:

Using problems from the introduction and Problem 14, find an expression for the coefficients of $F(x)$ (and this, for the Fibonacci numbers).

Solution

$$\begin{aligned} f_0 &= \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0 \\ f_1 &= \frac{1}{a\sqrt{5}} - \frac{1}{b\sqrt{5}} = 1 \\ f_n &= \frac{1}{\sqrt{5}} \left(\frac{1}{a^n} - \frac{1}{b^n} \right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \end{aligned}$$

Problem 16: Bonus

Repeat the method of recurrence, generating function, partial fraction decomposition, and geometric series to find a closed form for the following sequence:

$$a_0 = 1; \quad a_{n+1} = 2a_n + n$$

Hint: When doing partial fraction decomposition with a denominator of the form $(x-a)^2(x-b)$, you may need to express your expression as a sum of three fractions: $\frac{c}{(x-a)^2} + \frac{d}{x-a} + \frac{e}{x-b}$.

Part 3: Dice

Definition 17:

A *die* is a device that randomly selects a positive integer from a finite list of options. For example, the standard 6-sided die selects a value from $[1, 2, 3, 4, 5, 6]$. We may have many sides with the same value, as in $[1, 1, 2, 3]$.

To describe a die with a generating function, let a_k be the number of times k appears as a side of the die and consider $a_0 + a_1x + a_2x^2 + \dots$

A die has a finite number of sides, so this will be a regular polynomial.

Problem 18:

What is the generating function of the standard 6-sided die?

Solution

$$x + x^2 + x^3 + x^4 + x^5 + x^6$$

Problem 19:

What is the generating function of the die with sides $[1, 2, 3, 5]$?

Solution

$$2x + x^2 + x^3 + x^5$$

Problem 20:

Let $A(x)$ and $B(x)$ be the generating functions of two dice.

What is the significance of $A(1)$?

Solution

$$A(1) = \text{the number of sides on the die}$$

Problem 21:

Using formulas we found earlier, show that the k^{th} coefficient of $A(x)B(x)$ is the number of ways to roll k as the sum of the two dice.

Solution

The k^{th} coefficient of $A(x)B(x)$ is...

$$\begin{aligned} a_0b_k + a_1b_{k+1} + \dots + a_kb_0 \\ &= \text{count}(A = 0; B = k) + \dots + \text{count}(A = k; B = 0) \\ &= \text{number of ways } A + B = k \end{aligned}$$

Problem 22:

Find a generating function for the sequence c_0, c_1, \dots , where c_k is the probability that the sum of the two dice is k .

Solution

$$c_k = \frac{\text{number of ways sum} = k}{\text{number of total outcomes}} = \frac{\text{number of ways sum} = k}{A(1)B(1)}$$

So,

$$c_0 + c_1x + c_2x^2 = \frac{A(x)B(x)}{A(1)B(1)}$$

Problem 23:

Using generating functions, find two six-sided dice whose sum has the same distribution as the sum of two standard six-sided dice?

That is, for any integer k , the number of ways that the sum of the two nonstandard dice rolls as k is equal to the number of ways the sum of two standard dice rolls as k . *Hint: factor polynomials.*

Solution

We need a different factorization of

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = A(x)B(x)$$

We can use

$$(x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8)$$

Part 4: Coins

Consider the following problem:

“How many different ways can you make change for \$0.50 using pennies, nickels, dimes, quarters and half-dollars?”

Most ways of solving this involve awkward brute-force approaches that don't reveal anything interesting about the problem: how can we change our answer if we want to make change for \$0.51, or \$1.05, or some other quantity?

We can use generating functions to solve this problem in a general way.

Definition 24:

Let p_0, p_1, p_2, \dots be such that p_k is the number of ways to make change for k cents with only pennies. Similarly, let...

- n_k be the number of ways to make change for k cents with only nickels;
- d_k be the number of ways using only dimes;
- q_k be the number of ways using only quarters;
- and h_k be the number of ways using only half-dollars.

Problem 25:

Let $p(x)$ be the generating function that corresponds to p_n . Express $p(x)$ as a rational function.

Problem 26:

Modify Problem 25 to find expressions for $n(x)$, $d(x)$, $q(x)$, and $h(x)$.

Definition 27:

Now, let $N(x)$ be the generating function for the sequence n_0, n_1, \dots , where n_k is the number of ways to make change for k cents using pennies and nickels.

Similarly, let...

- let $D(x)$ be the generating function for the sequence using pennies, nickels, and dimes;
- let $Q(x)$ use pennies, nickels, dimes, and quarters;
- and let $H(x)$ use all coins.

Problem 28:

Express $N(x)$ as a rational function.

Problem 29:

Using the previous problem, write $D(x)$, then $Q(x)$, then $H(x)$ as rational functions.

Problem 30:

Using these generating functions, find recurrence relations for the sequences N_k , D_k , Q_k , and H_k .

Hint: Your recurrence relation for N_k should refer to the previous values of itself and some values of p_k . Your recurrence for D_k should refer to itself and N_k ; the one for Q_k should refer to itself D_k ; and the one for H_k should refer to itself and Q_k .

Problem 31:

Using these recurrence relations, fill following table and solve the original problem.

| n | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|-------|---|---|----|----|----|----|----|----|----|----|----|
| p_k | | | | | | | | | | | |
| N_k | | | | | | | | | | | |
| D_k | | | | | | | | | | | |
| Q_k | | | | | | | | | | | |
| H_k | | | | | | | | | | | |