

# Intro to Cryptography

Prepared by Mark on January 25, 2025

---

## Part 1: The Euclidean Algorithm

**Definition 1:**

The *greatest common divisor* of  $a$  and  $b$  is the greatest integer that divides both  $a$  and  $b$ . We denote this number with  $\gcd(a, b)$ . For example,  $\gcd(45, 60) = 15$ .

**Problem 2:**

Find  $\gcd(20, 14)$  by hand.

**Theorem 3: The Division Algorithm**

Given two integers  $a, b$ , we can find two integers  $q, r$ , where  $0 \leq r < b$  and  $a = qb + r$ . In other words, we can divide  $a$  by  $b$  to get  $q$  remainder  $r$ .

**Theorem 4:**

For any integers  $a, b, c$ ,  
 $\gcd(ac + b, a) = \gcd(a, b)$

**Problem 5: The Euclidean Algorithm**

Using the two theorems above, detail an algorithm for finding  $\gcd(a, b)$ . Then, compute  $\gcd(1610, 207)$  by hand.

**Problem 6:**

Using the output of the Euclidean algorithm,

- find a pair  $(u, v)$  that satisfies  $20u + 14v = \gcd(20, 14)$
- find a pair  $(u, v)$  that satisfies  $541u + 34v = \gcd(541, 34)$

This is called the *extended Euclidean algorithm*.

*Hint:* You don't need to fully solve the last part of this question.

Understand how you *would* do it, then move on. Don't spend too much time on arithmetic.

**Hint:**

After running the Euclidean algorithm, you have a table similar to the one shown below.

You can use a bit of algebra to rearrange these statements to get what you need.

Using the Euclidean Algorithm to find that  $\gcd(20, 14) = 2$ :

$$20 = 14 \times 1 + 6$$

$$14 = 6 \times 2 + 2$$

$$6 = 2 \times 3 + 0$$

We now want to write the 2 in the last equation in terms of 20 and 14.

## Part 2: Modular Arithmetic

### Definition 7:

$\mathbb{Z}_n$  is the set of integers mod  $n$ . For example,  $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ .

Multiplication in  $\mathbb{Z}_n$  works much like multiplication in  $\mathbb{Z}$ :

If  $a, b$  are elements of  $\mathbb{Z}_n$ ,  $a \times b$  is the remainder of  $a \times b$  when divided by  $n$ .

For example,  $2 \times 2 = 4$  and  $3 \times 4 = 12 = 2$  in  $\mathbb{Z}_5$

### Problem 8:

Create a multiplication table for  $\mathbb{Z}_4$ :

| $\times$ | 0 | 1 | 2 | 3 |
|----------|---|---|---|---|
| 0        | ? | ? | ? | ? |
| 1        | ? | ? | ? | ? |
| 2        | ? | ? | ? | ? |
| 3        | ? | ? | ? | ? |

### Definition 9:

Let  $a, b$  be elements of  $\mathbb{Z}_n$ . If  $a \times b = 1$ , we say that  $b$  is the *inverse* of  $a$  in  $\mathbb{Z}_n$ .

We usually write “ $a$  inverse” as  $a^{-1}$ .

Inverses are **not** guaranteed to exist.

### Theorem 10:

$a$  has an inverse in  $\mathbb{Z}_n$  if and only if  $\gcd(a, n) = 1$

### Problem 11:

Find the inverse of 3 in  $\mathbb{Z}_4$ , if one exists.

Find the inverse of 20 in  $\mathbb{Z}_{14}$ , if one exists.

Find the inverse of 4 in  $\mathbb{Z}_7$ , if one exists.

### Problem 12:

Show that if  $n$  is prime, every element of  $\mathbb{Z}_n$  (except 0) has an inverse.

### Problem 13:

Show that if  $n$  is not prime,  $\mathbb{Z}_n$  has at least one element with no inverse.

**Problem 14:**

In general, how can we find the inverse of  $a$  in  $\mathbb{Z}_n$ ? Assume  $a$  and  $n$  are coprime.

*Hint:* You can find that  $34^{-1}$  is  $-175$  in  $\mathbb{Z}_{541}$  by looking at a previous problem.

**Definition 15:**

Elements in  $\mathbb{Z}_n$  that have an inverse are called *units*.

The set of units in  $\mathbb{Z}_n$  is called  $\mathbb{Z}_n^\times$ , which is read “ $\mathbb{Z}$  mod  $n$  cross”.

**Problem 16:**

What is  $\mathbb{Z}_5^\times$ ?

What is  $\mathbb{Z}_{12}^\times$ ?

## Part 3: Groups

Group theory gives us a set of tools for understanding complex structures. We can use groups to solve the Rubik's cube, to solve problems in physics and chemistry, and to understand complex geometric symmetries. It's also worth noting that much of modern cryptography is built using results from group theory.

### Definition 17:

A *group*  $(G, *)$  consists of a set  $G$  and an operator  $*$ .

Groups always have the following properties:

**A:**  $G$  is closed under  $*$ . In other words,  $a, b \in G \implies a * b \in G$ .

**B:**  $*$  is associative:  $(a * b) * c = a * (b * c)$  for all  $a, b, c \in G$

**C:** There is an *identity*  $e \in G$ , so that  $a * e = a * e = a$  for all  $a \in G$ .

**D:** For any  $a \in G$ , there exists a  $b \in G$  so that  $a * b = b * a = e$ .  $b$  is called the *inverse* of  $a$ .

This element is written as  $-a$  if our operator is addition and  $a^{-1}$  otherwise.

Any pair  $(G, *)$  that satisfies these properties is a group.

### Problem 18:

Is  $(\mathbb{Z}_5, +)$  a group?

Is  $(\mathbb{Z}_5, -)$  a group?

*Hint:*  $+$  and  $-$  refer to the usual operations in modular arithmetic.

### Problem 19:

Show that  $(\mathbb{R}, \times)$  is not a group, then find a subset  $S$  of  $\mathbb{R}$  so that  $(S, \times)$  is a group.

### Problem 20:

What is the smallest group we can create?

**Problem 21:**

Let  $(G, *)$  be a group with finitely many elements, and let  $a \in G$ .

Show that there exists an  $n$  in  $\mathbb{Z}^+$  so that  $a^n = e$

*Hint:*  $a^n := a * a * \dots * a$ , with  $a$  repeated  $n$  times.

The smallest such  $n$  defines the *order* of  $g$ .

**Problem 22:**

What is the order of 5 in  $(\mathbb{Z}_{25}, +)$ ?

What is the order of 2 in  $(\mathbb{Z}_{17}^\times, \times)$ ?

**Theorem 23:**

Let  $p$  be a prime number.

In any group  $(\mathbb{Z}_p^\times, *)$  there exists a  $g \in \mathbb{Z}_p^\times$  where...

- The order of  $g$  is  $p - 1$ , and
- $\{a^0, a^1, \dots, a^{p-2}\} = \mathbb{Z}_p^\times$

We call such a  $g$  a *generator*, since its powers generate every other element in the group.

## Part 4: The Discrete Log Problem

### Definition 24:

Let  $g$  be a generator in  $(\mathbb{Z}_p^\times, *)$

Let  $n$  be a positive integer.

We now want a function “log” from  $\mathbb{Z}_p^\times$  to  $\mathbb{Z}^+$  so that  $\log_g(g^n) = n$ .

In other words, we want an inverse of the “exponent” function.

This is the *discrete logarithm problem*, often abbreviated *DLP*.

### Problem 25:

Does the discrete log function even exist?

Show that exp is a bijection, which will guarantee the existence of log.

*Note:* Why does this guarantee the existence of log? Recall our lesson on functions.

### Problem 26:

Find a simple (but perhaps inefficient) way to calculate  $\log_g(a)$

### Problem 27:

Find an efficient way to solve the discrete log problem.

Then learn L<sup>A</sup>T<sub>E</sub>X, write a paper, and enjoy free admission to the graduate program at any university.

The discrete logarithm can be quickly computed in a few special cases, but there is no known way to efficiently compute it in general. Interestingly enough, we haven’t been able to prove that an efficient solution *doesn’t* exist. The best we can offer is a “proof by effort:” many smart people have been trying for long time and haven’t solved it yet. It probably doesn’t exist.

In the next few pages, we’ll see how the assumption “DLP is hard” can be used to construct various tools used to secure communications.

## Part 5: Diffie-Hellman Key Exchange

One problem we encounter in computer science is *secure key exchange*: How can two parties (usually called Alice and Bob) agree on a “key” without revealing anything to an eavesdropper (Eve)?



A simple mathematical solution to the key exchange problem is the *Diffie-Hellman key exchange algorithm*, detailed below.

Values that are *public* are known to everyone. Values that are sent are also known to everyone: we assume that everyone can see what Alice and Bob send to each other.

Eve can read all public values, but she cannot change them in any way.

### Setup

Let  $p$  be a prime number  
 Let  $g$  be a generator in  $\mathbb{Z}_p^\times$   
 Both  $g$  and  $p$  are public.

| Alice  | Public | Bob  |
|--|--------|--|
| Pick a random $a \in \mathbb{Z}_p^\times$<br>Set $A = g^a$ | $p, g$ | Pick a random $b \in \mathbb{Z}_p^\times$<br>Set $B = g^b$ |
| Publish $A$  | $A$    |  |
|  | $B$    | Publish $B$  |
| Compute ...  |        | Compute ...  |

### Problem 28:

Complete the algorithm. What should Alice and Bob compute?

*Hint:* The goal of this process is to arrive at a *shared secret*

That is, Alice and Bob should arrive at the same value without exposing it to Eve.

### Problem 29:

Let  $p = 11$ ,  $g = 2$ ,  $a = 9$ , and  $b = 4$ .

Run the algorithm. What is the resulting shared secret?



**Problem 30:**

Is the Diffie-Hellman key exchange algorithm secure? What information does Eve have? What does Eve need to do to find the value Alice and Bob agreed on?

**Problem 31:**

Now, say Eve can change information in transit.

That is, she can pretend to be Alice to send information to Bob.

How can she break this system?

*Note:* This is called a *man-in-the-middle* attack.

## Part 6: Elgamal Asymmetric Encryption

Another cryptographic tool we often use is the *public key cryptosystem*. In such a system, one has two keys: a *public key* that can only encrypt data, and a *private key* that can decrypt it. The following problem provides a simple example.

### Problem 32:

Alice wants to send a secret letter to Bob. Eve, the postman, would like to see what is inside.

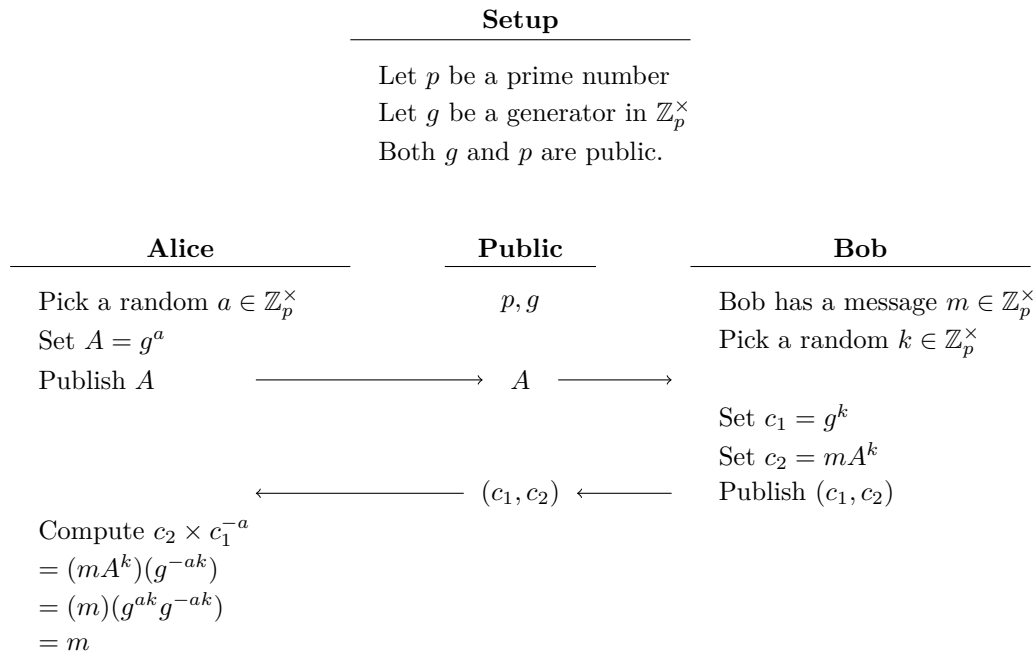
Alice has a box, a lock, and a key. Bob does not own a lock.

Eve will open the box if she can, but she will not try to break any locks.

Also, she will always deliver the box without modifying its contents.

How can Alice send her letter without letting Eve read it?

Elgamal encryption allows Alice to publish a public key ( $A$  in the diagram below), which Bob can use to encrypt a message. Alice then uses her private key ( $a$ ) to decrypt it.



### Problem 33:

Let  $p = 17$ ,  $g = 2$ ,  $a = 7$ ,  $k = 10$ , and  $m = 3$

Run this algorithm and make sure it works.

**Problem 34:**

What information does Eve have?

What does Eve need to do to find  $m$ ?

**Problem 35:**

Say Bob re-uses the same  $k$  twice.

Let  $(c_1, c_2)$  and  $(d_1, d_2)$  be two ciphertexts generated with this key, encrypting messages  $m_1$  and  $m_2$ .

Also, say Eve knows the value of  $m_1 - m_2$ . How can Eve find  $m_1$  and  $m_2$ ?

*Note:* If Bob doesn't change his key, Eve will also be able to decrypt future messages.

## Part 7: Bonus Problems

**Problem 36:**

Show that a group has exactly one identity element.

**Problem 37:**

Show that each element in a group has exactly one inverse.

**Problem 38:**

Let  $(G, *)$  be a group and  $a, b, c \in G$ . Show that...

- $a * b = a * c \implies b = c$
- $b * a = c * a \implies b = c$

This means that we can “cancel” operations in groups, much like we do in algebra.

**Problem 39:**

Let  $G$  be the set of all bijections  $A \rightarrow A$ .

Let  $\circ$  be the usual composition operator.

Is  $(G, \circ)$  a group?

**Definition 40:**

Note that our definition of a group does **not** state that  $a * b = b * a$ .

Many interesting groups do not have this property. Those that do are called *abelian* groups.

One example of a non-abelian group is the set of invertible 2x2 matrices under matrix multiplication.

**Problem 41:**

Show that if  $G$  has four elements,  $(G, *)$  is abelian.

**Problem 42:**

Prove ??:

$a$  has an inverse mod  $m$  iff  $\gcd(a, m) = 1$

**Problem 43:**

The Euclidean Algorithm (From ??) can be written as follows:

- Assume  $a > b$ .
- Set  $e_0 = a$  and  $e_1 = b$ .
- Let  $e_{n+1} = \text{remainder}(e_{n-1} \div e_n)$
- Stop when  $e_k = 0$ .
- Then,  $\gcd(a, b) = e_{k-1}$ .

Let  $F_n$  be the  $n^{\text{th}}$  Fibonacci number. ( $F_0 = 0$ ;  $F_1 = 1$ ;  $F_2 = 1$ ;  $\dots$ )

Show that if the Euclidean algorithm requires  $n$  steps for an input  $(a, b)$ , then  $a \geq F_{n+2}$  and  $b \geq F_{n+1}$ . In other words, show that the longest-running input of a given size is a Fibonacci pair.

**Problem 44: Chinese Remainder Theorem**

There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?

**Problem 45:**

Show that if  $p$  is prime,  $\binom{p}{i} \equiv 0 \pmod{p}$  for  $0 < i < p$ .

**Problem 46: Fermat's Little Theorem**

Show that if  $p$  is prime and  $a \not\equiv 0 \pmod{p}$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .

You may want to use ??.

*Hint:* It may be easier to show that  $a^p \equiv a \pmod{p}$

**Problem 47:**

Show that for any three integers  $a, b, c$ ,

$$\gcd(ac + b, a) = \gcd(a, b)$$

[Note on ??] This proof can be used to show that the Euclidean algorithm finishes in logarithmic time, and it is the first practical application of the Fibonacci numbers. If you have finished all challenge problems, finish the proof: find how many steps the Euclidean algorithm needs to arrive at a solution for a given  $a$  and  $b$ .