

# To Mock a Mockingbird

Prepared by Mark on January 27, 2025  
Based on a book of the same name.

## Part 1: Introduction

A certain enchanted forest is inhabited by talking birds. Each of these birds has a name, and will respond whenever it hears the name of another. Suppose you are exploring this forest and come across the bird  $A$ . You call the name of bird  $B$ .  $A$  hears you and responds with the name of some other bird, which we will designate  $AB$ .

Bird  $AB$  is, by definition,  $A$ 's response to  $B$ .

As you wander around this forest, you quickly discover two interesting facts:

**A:**  $A$ 's response to  $B$  mustn't be the same as  $B$ 's response to  $A$ .

**B:** Given three birds  $A$ ,  $B$ , and  $C$ ,  $(AB)C$  and  $A(BC)$  are not necessarily the same bird.

Bird  $A(BC)$  is  $A$ 's response to bird  $BC$ , while  $(AB)C$  is  $AB$ 's response to  $C$ .

Thus,  $ABC$  is ambiguous. Parenthesis are mandatory.

You also find that this forest has two laws:

**A:** *The Law of Composition:*

For any two birds  $A$  and  $B$ , there must be a bird  $C$  so that  $Cx = A(Bx)$

**B:** *The Law of the Mockingbird:*

The forest must contain the Mockingbird  $M$ , which always satisfies  $Mx = xx$ .

In other words, the Mockingbird's response to any bird  $x$  is the same as  $x$ 's response to itself.

### Definition 1:

We say a bird  $A$  is fond of a bird  $B$  if  $A$  responds to  $B$  with  $B$ .

In other words,  $A$  is fond of  $B$  if  $AB = B$ .

### Definition 2:

We say a bird  $C$  composes  $A$  with  $B$  if for any bird  $x$ ,

$$Cx = A(Bx)$$

In other words, this means that  $C$ 's response to  $x$  is the same as  $A$ 's response to  $B$ 's response to  $x$ .

Note that  $C$  is exactly the kind of bird  $L_1$  guarantees.

## Part 2: To Mock a Mockingbird

### Problem 3:

Mark tells you that any bird  $A$  is fond of at least one other bird.

Complete his proof.

```
let A          # Let A be any any bird.
let Cx = A(Mx) # Define C as the composition of A and M

# The rest is up to you.
CC = ??
```

#### Things you will need:

**Law:** There exists a Mockingbird,  $Mx := xx$

**Def:**  $A$  is fond of  $B$  if  $AB = B$

### Problem 4:

We say a bird  $A$  is *egocentric* if it is fond of itself.

Show that the laws of the forest guarantee that at least one bird is egocentric.

#### Things you will need:

**Law:** There exists a Mockingbird,  $Mx := xx$

**Def:**  $A$  is fond of  $B$  if  $AB = B$

**Lem:** Any bird is fond of at least one bird.

**Definition 5:**

We say a bird  $A$  is *agreeable* if for all birds  $B$ , there is at least one bird  $x$  on which  $A$  and  $B$  agree. In other words,  $A$  is agreeable if given any  $B$ , we can find a bird  $x$  satisfying  $Ax = Bx$ .

**Problem 6:**

Is the Mockingbird agreeable?

**Problem 7:**

Take two birds  $A$  and  $B$ . Let  $C$  be their composition.

Show that if  $C$  is agreeable,  $A$  is agreeable.

```
# Given information
```

```
let A, B
```

```
let Cx = A(Bx)
```

```
let D                # Arbitrary bird
```

```
let Ex = D(Bx)      # Define E as the composition of D and B
```

```
Cy = ??
```

**Things you will need:**

**Def:**  $A$  is agreeable if  $Ax = Bx$  for all  $B$  with some  $x$ .

**Law:** For any  $A, B$ , there is  $C$  defined by  $Cx = A(Bx)$

**Problem 8:**

Given three arbitrary birds  $A$ ,  $B$ , and  $C$ , show that there exists a bird  $D$  satisfying  $Dx = A(B(Cx))$

**Definition 9:**

We say two birds  $A$  and  $B$  are *compatible* if there are birds  $x$  and  $y$  so that  $Ax = y$  and  $By = x$ . Note that  $x$  and  $y$  may be the same bird.

**Problem 10:**

Show that any two birds in this forest are compatible.

```
let A, B
let Cx = A(Bx)
```

**Things you will need:**

**Law:** Law of composition

**Lem:** Any bird is fond of at least one bird.

**Problem 11:**

Show that any bird that is fond of at least one bird is compatible with itself.

## Part 3: The Curious Kestrel

**Definition 12:**

Recall that a bird is *egocentric* if it is fond of itself.

A bird is *hopelessly egocentric* if  $Bx = B$  for all birds  $x$ .

**Definition 13:**

More generally, we say that a bird  $A$  is *fixated* on a bird  $B$  if  $Ax = B$  for all  $x$ .

Convince yourself that a hopelessly egocentric bird is fixated on itself.

**Problem 14:**

Say  $A$  is fixated on  $B$ . Is  $A$  fond of  $B$ ?

**Definition 15:**

The *Kestrel*  $K$  is defined by the following relationship:

$$(Kx)y = x \quad \forall x, y$$

In other words, this means that for every bird  $x$ , the bird  $Kx$  is fixated on  $x$ .

**Problem 16:**

Show that an egocentric Kestrel is hopelessly egocentric.

**Problem 17:**

Assume the forest contains a Kestrel.

Given the Law of Composition and the Law of the Mockingbird, show that at least one bird is hopelessly egocentric.

**Things you will need:**

Def:  $K$  is defined by  $(Kx)y = x$

Def:  $A$  is fond of  $B$  if  $AB = B$

???: You'll need one more result from the previous section. Good luck!

**Problem 18: Kestrel Left-Cancellation**

In general,  $Ax = Ay$  does not imply  $x = y$ . However, this is true if  $A$  is  $K$ .

Show that  $Kx = Ky \implies x = y$ .

# This is a hint.

let  $x, y$  so that  $Kx = Ky$

**Problem 19:**

Show that if  $K$  is fond of  $Kx$ ,  $K$  is fond of  $x$ .

**Problem 20:**

An egocentric Kestrel must be extremely lonely. Why is this?