

# Generating Functions

Prepared by Mark on January 27, 2025  
Based on a handout by Aaron Anderson

## Instructor's Handout

This file contains solutions and notes.  
Compile with the “nosolutions” flag before distributing.  
Click [\[here\]](#) for the latest version of this handout.

## Part 1: Introduction

### Definition 1:

Say we have a sequence  $a_0, a_1, a_2, \dots$

The *generating function* of this sequence is defined as follows:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Under some circumstances, this sum does not converge, and thus  $A(x)$  is undefined.

However, we can still manipulate this infinite sum to get useful results even if  $A(x)$  diverges.

### Problem 2:

Let  $A(x)$  be the generating function of the sequence  $a_n$ ,

and let  $B(x)$  be the generating function of the sequence  $b_n$ .

Find the sequences that correspond to the following generating functions:

- $cA(x)$
- $xA(x)$
- $A(x) + B(x)$
- $A(x)B(x)$

### Solution

- $cA(x)$  corresponds to  $ca_n$
- $xA(x)$  corresponds to  $0, a_0, a_1, \dots$
- $A(x) + B(x)$  corresponds to  $a_n + b_n$
- $A(x)B(x)$  is  $a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots$   
Which corresponds to  $c_n = \sum_{k=0}^n a_k b_{n-k}$

**Problem 3:**

Assuming  $|x| < 1$ , show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

*Hint:* use some clever algebra. What is  $x \times (1 + x + x^2 + \dots)$ ?

**Solution**

Let  $S = 1 + x + x^2 + \dots$   
 Then,  $xS = x + x^2 + x^3 + \dots$   
 So,  $xS = S - 1$   
 and  $1 = S - xS = S(1 - x)$   
 and  $S = \frac{1}{1-x}$ .

**Problem 4:**

Let  $A(x)$  be the generating function of the sequence  $a_n$ .

Find the sequence that corresponds to the generating function  $\frac{A(x)}{1-x}$

**Solution**

$$\begin{aligned} \frac{A(x)}{1-x} &= A(x)(1 + x + x^2 + \dots) \\ &= (a_0 + a_1x + a_2x^2 + \dots)(1 + x + x^2 + \dots) \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots \end{aligned}$$

Which corresponds to the sequence  $c_n = \sum_{k=0}^n a_k$

**Problem 5:**

Find short expressions for the generating functions for the following sequences:

- 1, 0, 1, 0, ...
- 1, 2, 4, 8, 16, ...
- 1, 2, 3, 4, 5, ...

**Solution**

- 1, 0, 1, 0, ... corresponds to  $1 + x^2 + x^4 + \dots$   
 By Problem 3, this is  $\frac{1}{1-x^2}$ .
- 1, 2, 4, 8, 16, ... corresponds to  $1 + 2x + (2x)^2 + \dots$   
 By Problem 3, this is  $\frac{1}{1-2x}$ .
- 1, 2, 3, 4, 5, ... corresponds to  $1 + 2x + 3x^2 + 4x^3 + \dots$   
 This is equal to  $(1 + x + x^2 + \dots)^2$ , and thus is  $\left(\frac{1}{1-x}\right)^2$

## Part 2: Fibonacci

**Definition 6:**

The *Fibonacci numbers* are defined by the following recurrence relation:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$

**Problem 7:**

Let  $F(x)$  be the generating function that corresponds to the Fibonacci numbers.

Find the generating function of  $0, f_0, f_1, \dots$  in terms of  $F(x)$ .

Call this  $G(x)$ .

**Solution**

$$G(x) = xF(x)$$

**Problem 8:**

Find the generating function of  $0, 0, f_0, f_1, \dots$  in terms of  $F(x)$ . Call this  $H(x)$ .

**Solution**

$$H(x) = x^2F(x)$$

**Problem 9:**

Calculate  $F(x) - G(x) - H(x)$  using the recurrence relation that we used to define the Fibonacci numbers.

**Solution**

$$\begin{aligned} F(x) - G(x) - H(x) &= f_0 + (f_1 - f_0)x + (f_2 - f_1 - f_0)x^2 + (f_3 - f_2 - f_1)x^3 + \dots \\ &= f_0 + (f_1 - f_0)x \\ &= x \end{aligned}$$

**Problem 10:**

Using the problems on the previous page, find  $F(x)$  in terms of  $x$ .

**Solution**

$$\begin{aligned} x &= F(x) - G(x) - H(x) \\ &= F(x) - xF(x) - x^2F(x) \\ &= F(x)(1 - x - x^2) \end{aligned}$$

So,

$$F(x) = \frac{x}{1 - x - x^2}$$

**Definition 11:**

A *rational function*  $f$  is a function that can be written as a quotient of polynomials.

That is,  $f(x) = \frac{p(x)}{q(x)}$  where  $p$  and  $q$  are polynomials.

**Problem 12:**

Solve the equation from ??fib> for  $F(x)$ , expressing it as a rational function.

**Solution**

$$\begin{aligned} F(x) &= \frac{-x}{x^2 + x - 1} = \frac{-x}{(x - a)(x - b)} \\ &= \frac{1 - \sqrt{5}}{2\sqrt{5}} \frac{1}{x - a} + \frac{-1 - \sqrt{5}}{2\sqrt{5}} \frac{1}{x - b} \end{aligned}$$

where

$$a = \frac{-1 + \sqrt{5}}{2}; \quad b = \frac{-1 - \sqrt{5}}{2}$$

**Definition 13:**

*Partial fraction decomposition* is an algebraic technique that works as follows:

If  $p(x)$  is a polynomial and  $a$  and  $b$  are constants, we can rewrite the rational function  $\frac{p(x)}{(x-a)(x-b)}$  as follows:

$$\frac{p(x)}{(x-a)(x-b)} = \frac{c}{x-a} + \frac{d}{x-b}$$

where  $c$  and  $d$  are constants.

**Problem 14:**

Now that we have a rational function for  $F(x)$ ,

find a closed-form expression for its coefficients using partial fraction decomposition.

**Solution**

$$\begin{aligned} F(x) &= \left( \frac{1-\sqrt{5}}{2\sqrt{5}} \right) \left( \frac{-1}{a} \right) \left( \frac{1}{1-\frac{x}{a}} \right) + \left( \frac{-1-\sqrt{5}}{2\sqrt{5}} \right) \left( \frac{-1}{b} \right) \left( \frac{1}{1-\frac{x}{b}} \right) \\ &= \left( \frac{1}{\sqrt{5}} \right) \left( \frac{1}{1-\frac{x}{a}} \right) + \left( \frac{-1}{\sqrt{5}} \right) \left( \frac{1}{1-\frac{x}{b}} \right) \\ &= \frac{1}{\sqrt{5}} \left( 1 + \frac{x}{a} + \left( \frac{x}{a} \right)^2 + \dots \right) - \frac{1}{\sqrt{5}} \left( 1 + \frac{x}{b} + \left( \frac{x}{b} \right)^2 + \dots \right) \end{aligned}$$

**Problem 15:**

Using problems from the introduction and Problem 14, find an expression for the coefficients of  $F(x)$  (and this, for the Fibonacci numbers).

**Solution**

$$\begin{aligned} f_0 &= \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0 \\ f_1 &= \frac{1}{a\sqrt{5}} - \frac{1}{b\sqrt{5}} = 1 \\ f_n &= \frac{1}{\sqrt{5}} \left( \frac{1}{a^n} - \frac{1}{b^n} \right) = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right) \end{aligned}$$

**Problem 16: Bonus**

Repeat the method of recurrence, generating function, partial fraction decomposition, and geometric series to find a closed form for the following sequence:

$$a_0 = 1; \quad a_{n+1} = 2a_n + n$$

*Hint:* When doing partial fraction decomposition with a denominator of the form  $(x-a)^2(x-b)$ , you may need to express your expression as a sum of three fractions:  $\frac{c}{(x-a)^2} + \frac{d}{x-a} + \frac{e}{x-b}$ .

## Part 3: Dice

**Definition 17:**

A *die* is a device that randomly selects a positive integer from a finite list of options. For example, the standard 6-sided die selects a value from  $[1, 2, 3, 4, 5, 6]$ . We may have many sides with the same value, as in  $[1, 1, 2, 3]$ .

To describe a die with a generating function, let  $a_k$  be the number of times  $k$  appears as a side of the die and consider  $a_0 + a_1x + a_2x^2 + \dots$

A die has a finite number of sides, so this will be a regular polynomial.

**Problem 18:**

What is the generating function of the standard 6-sided die?

**Solution**

$$x + x^2 + x^3 + x^4 + x^5 + x^6$$

**Problem 19:**

What is the generating function of the die with sides  $[1, 2, 3, 5]$ ?

**Solution**

$$2x + x^2 + x^3 + x^5$$

**Problem 20:**

Let  $A(x)$  and  $B(x)$  be the generating functions of two dice.

What is the significance of  $A(1)$ ?

**Solution**

$$A(1) = \text{the number of sides on the die}$$

**Problem 21:**

Using formulas we found earlier, show that the  $k^{\text{th}}$  coefficient of  $A(x)B(x)$  is the number of ways to roll  $k$  as the sum of the two dice.

**Solution**

The  $k^{\text{th}}$  coefficient of  $A(x)B(x)$  is...

$$\begin{aligned} a_0b_k + a_1b_{k+1} + \dots + a_kb_0 \\ &= \text{count}(A = 0; B = k) + \dots + \text{count}(A = k; B = 0) \\ &= \text{number of ways } A + B = k \end{aligned}$$

**Problem 22:**

Find a generating function for the sequence  $c_0, c_1, \dots$ , where  $c_k$  is the probability that the sum of the two dice is  $k$ .

**Solution**

$$c_k = \frac{\text{number of ways sum} = k}{\text{number of total outcomes}} = \frac{\text{number of ways sum} = k}{A(1)B(1)}$$

So,

$$c_0 + c_1x + c_2x^2 = \frac{A(x)B(x)}{A(1)B(1)}$$

**Problem 23:**

Using generating functions, find two six-sided dice whose sum has the same distribution as the sum of two standard six-sided dice?

That is, for any integer  $k$ , the number of ways that the sum of the two nonstandard dice rolls as  $k$  is equal to the number of ways the sum of two standard dice rolls as  $k$ . *Hint: factor polynomials.*

**Solution**

We need a different factorization of

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = A(x)B(x)$$

We can use

$$(x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8)$$



## Part 4: Coins

Consider the following problem:

“How many different ways can you make change for \$0.50 using pennies, nickels, dimes, quarters and half-dollars?”

Most ways of solving this involve awkward brute-force approaches that don't reveal anything interesting about the problem: how can we change our answer if we want to make change for \$0.51, or \$1.05, or some other quantity?

We can use generating functions to solve this problem in a general way.

### Definition 24:

Let  $p_0, p_1, p_2, \dots$  be such that  $p_k$  is the number of ways to make change for  $k$  cents with only pennies. Similarly, let...

- $n_k$  be the number of ways to make change for  $k$  cents with only nickels;
- $d_k$  be the number of ways using only dimes;
- $q_k$  be the number of ways using only quarters;
- and  $h_k$  be the number of ways using only half-dollars.

### Problem 25:

Let  $p(x)$  be the generating function that corresponds to  $p_n$ . Express  $p(x)$  as a rational function.

### Problem 26:

Modify Problem 25 to find expressions for  $n(x)$ ,  $d(x)$ ,  $q(x)$ , and  $h(x)$ .

**Definition 27:**

Now, let  $N(x)$  be the generating function for the sequence  $n_0, n_1, \dots$ , where  $n_k$  is the number of ways to make change for  $k$  cents using pennies and nickels.

Similarly, let...

- let  $D(x)$  be the generating function for the sequence using pennies, nickels, and dimes;
- let  $Q(x)$  use pennies, nickels, dimes, and quarters;
- and let  $H(x)$  use all coins.

**Problem 28:**

Express  $N(x)$  as a rational function.

**Problem 29:**

Using the previous problem, write  $D(x)$ , then  $Q(x)$ , then  $H(x)$  as rational functions.

**Problem 30:**

Using these generating functions, find recurrence relations for the sequences  $N_k$ ,  $D_k$ ,  $Q_k$ , and  $H_k$ .

*Hint:* Your recurrence relation for  $N_k$  should refer to the previous values of itself and some values of  $p_k$ . Your recurrence for  $D_k$  should refer to itself and  $N_k$ ; the one for  $Q_k$  should refer to itself  $D_k$ ; and the one for  $H_k$  should refer to itself and  $Q_k$ .

**Problem 31:**

Using these recurrence relations, fill following table and solve the original problem.

$n$	0	5	10	15	20	25	30	35	40	45	50
$p_k$											
$N_k$											
$D_k$											
$Q_k$											
$H_k$											