

Intro to Proofs

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Part 1:

Problem 1:

We say an integer x is *even* if $x = 2k$ for some $k \in \mathbb{Z}$. We say x is *odd* if $x = 2k + 1$ for some $k \in \mathbb{Z}$. Assume that every integer is even or odd, and never both.

- Show that the product of two odd integers is odd.
- Let $a, b \in \mathbb{Z}, a \neq 0$. We say a *divides* b and write $a \mid b$ if there is a $k \in \mathbb{Z}$ so that $ak = b$. Show that $a \mid b \implies a \mid 2b$
- Show that $5n^2 + 3n + 7$ is odd for any $n \in \mathbb{Z}$.
- Let a, b, c be integers so that $a^2 + b^2 = c^2$. Show that one of a, b is even.
- Show that every odd integer is the difference of two squares.
- Prove the assumption in the statement of this problem.

Problem 2:

Let $r \in \mathbb{R}$. We say r is *rational* if there exist $p, q \in \mathbb{Z}, q \neq 0$ so that $r = \frac{p}{q}$

- Show that $\sqrt{2}$ is irrational.
- Show that the product of two rational numbers must be rational, while the product of irrational numbers may be rational or irrational. If you claim a number is irrational, provide a proof.

Problem 3:

Let $X = \{x \in \mathbb{Z} \mid x \geq 2\}$. For $k \geq 2$, define $X_k = \{kx \mid x \in X\}$. What is $X - (X_2 \cup X_3 \cup X_4 \cup \dots)$? Prove your claim.

Problem 4:

Show that there are infinitely many primes.

You may use the fact that every integer has a prime factorization.

Problem 5:

For a set X , define its *diagonal* as $D(X) = \{(x, x) \in X \times X \mid x \in X\}$.

An *undirected graph* G is an ordered pair (V, E) , where V is a set, and $E \subset V \times V$ satisfies $(a, b) \in E \iff (b, a) \in E$ and $E \cap D(X) = \emptyset$.

The elements of V are called *vertices*; the elements of E are called *edges*.

- Make sense of the conditions on E .
- The *degree* of a vertex a is the number of edges connected to that vertex.
We'll denote this as $d(a)$. Write a formal definition of this function using set-builder notation and the definitions above. Recall that $|X|$ denotes the size of a set X .
- There are 9 people at a party. Show that they cannot each have 3 friends.
Friendship is always mutual.

Problem 6:

Let f be a function from a set X to a set Y . We say f is *injective* if $f(x) = f(y) \implies x = y$.

We say f is *surjective* if for all $y \in Y$ there exists an $x \in X$ so that $f(x) = y$.

Let A, B, C be sets, and let $f : A \rightarrow B$, $g : B \rightarrow C$ be functions. Let $h = g \circ f$.

- Show that if h is injective, f must be injective and g may not be injective.
- Show that if h is surjective, g must be surjective and f may not be surjective.

Problem 7:

Let $X = \{1, 2, \dots, n\}$ for some $n \geq 2$. Let $k \in \mathbb{Z}$ so that $1 \leq k \leq n - 1$.

Let $E = \{Y \subset X \mid |Y| = k\}$, $E_1 = \{Y \in E \mid 1 \in Y\}$, and $E_2 = \{Y \in E \mid 1 \notin Y\}$

- Show that $\{E_1, E_2\}$ is a partition of E .
In other words, show that $\emptyset \neq E_1$, $\emptyset \neq E_2$, $E_1 \cup E_2 = E$, and $E_1 \cap E_2 = \emptyset$.
Hint: What does this mean in English?
- Compute $|E_1|$, $|E_2|$, and $|E|$.
Recall that a set of size n has $\binom{n}{k}$ subsets of size k .
- Conclude that for any n and k satisfying the conditions above,

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

- For $t \in \mathbb{N}$, show that $\binom{2t}{t}$ is even.

Theorem 8: The Division Algorithm

Given two integers a, b , we can find two integers q, r , where $0 \leq r < b$ and $a = qb + r$. In other words, we can divide a by b to get q remainder r .

Problem 9:

Let $x, y \in \mathbb{N}$ be natural numbers. Consider the set $S = \{ax + by \mid a, b \in \mathbb{Z}, ax + by > 0\}$.

The well-ordering principle states that every nonempty subset of the natural numbers has a least element.

- Show that S has a least element. Call it d .
- Let $z = \gcd(x, y)$. Show that z divides d .
- Show that d divides x and d divides y .
- Prove or disprove $\gcd(x, y) \in S$.

Problem 10:

- Let $f : X \rightarrow Y$ be an injective function. Show that for any two functions $g : Z \rightarrow X$ and $h : Z \rightarrow X$, if $f \circ g = f \circ h$ from Z to Y then $g = h$ from Z to X .

By definition, functions are equal if they agree on every input in their domain.

Hint: This is a one-line proof.

- Let $f : X \rightarrow Y$ be a surjective function. Show that for any two functions $g : Y \rightarrow W$ and $h : Y \rightarrow W$, if $g \circ f = h \circ f \implies g = h$.
- ★ Let $f : X \rightarrow Y$ be a function where for any set Z and functions $g : Z \rightarrow X$ and $h : Z \rightarrow X$, $f \circ g = f \circ h \implies g = h$. Show that f is injective.
- ★ Let $f : X \rightarrow Y$ be a function where for any set W and functions $g : Y \rightarrow W$ and $h : Y \rightarrow W$, $g \circ f = h \circ f \implies g = h$. Show f is surjective.

Problem 11:

In this problem we prove the binomial theorem: for $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, we have

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

In the proof below, we let a and b be arbitrary numbers.

- Check that this formula works for $n = 0$. Also, check a few small n to get a sense of what's going on.
- Let $N \in \mathbb{N}$. Suppose we know that for a specific value of N ,

$$(a + b)^N = \sum_{k=0}^N \binom{N}{k} a^k b^{N-k}$$

Now, show that this formula also works for $N + 1$.

- Conclude that this formula works for all $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Problem 12:

A *relation* on a set X is an $R \subset X \times X$.

- We say R is *reflexive* if $(x, x) \in R$ for all $x \in X$.
- We say R is *symmetric* if $(x, y) \in R \implies (y, x) \in R$.
- We say R is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.
- We say R is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Say we have a set X and an equivalence relation R .

The *equivalence class* of an element $x \in X$ is the set $\{y \in X \mid (x, y) \in R\}$.

Let R be an equivalence relation on a set X .

Show that the set of equivalence classes is a partition of X .

Problem 13:

Show that there exist two positive irrational numbers a and b so that a^b is rational.

Problem 14:

Show that the following holds for any planar graph:

$$\text{vertices} - \text{edges} + \text{faces} = 2$$

Hint: If you don't know what these words mean, ask an instructor.

Problem 15:

Consider a rectangular chocolate bar of arbitrary size.

What is the minimum number of breaks you need to make to separate all its pieces?

Problem 16:

Four travellers are on a plane, each moving along a straight line at an arbitrary constant speed.

No two of their paths are parallel, and no three intersect at the same point.

We know that traveller A has met traveler B, C, and D, and that B has met C and D (and A).

Show that C and D must also have met.

Problem 17:

Say we have an n -gon with non-intersecting edges.

What is the size of the smallest set of vertices that can “see” every point inside the polygon?