

Continued Fractions

Prepared by Mark on February 6, 2025

Based on a handout by Matthew Gherman and Adam Lott

Instructor's Handout

Part 1: The Euclidean Algorithm

Definition 1:

The *greatest common divisor* of a and b is the greatest integer that divides both a and b . We denote this number with $\gcd(a, b)$. For example, $\gcd(45, 60) = 15$.

Problem 2:

Find $\gcd(20, 14)$ by hand.

Solution

$$\gcd(20, 14) = 2$$

Theorem 3: The Division Algorithm

Given two integers a, b , we can find two integers q, r , where $0 \leq r < b$ and $a = qb + r$. In other words, we can divide a by b to get q remainder r .

For example, take $14 \div 3$. We can re-write this as $3 \times 4 + 2$.

Here, a and b are 14 and 3, $q = 4$ and $r = 2$.

Theorem 4:

For any integers a, b, c ,
 $\gcd(ac + b, a) = \gcd(a, b)$

Problem 5:

Compute $\gcd(668, 6)$. *Hint:* $668 = 111 \times 6 + 2$

Then, compute $\gcd(3 \times 668 + 6, 668)$.

Problem 6: The Euclidean Algorithm

Using the two theorems above, detail an algorithm for finding $\gcd(a, b)$.

Then, compute $\gcd(1610, 207)$ by hand.

Solution

Using Theorem 4 and the division algorithm,

$\gcd(1610, 207)$	$1610 = 207 \times 7 + 161$
$= \gcd(207, 161)$	$207 = 161 \times 1 + 46$
$= \gcd(161, 46)$	$161 = 46 \times 3 + 23$
$= \gcd(46, 23)$	$46 = 23 \times 2 + 0$
$= \gcd(23, 0) = 23$	

Part 2:

Definition 7:

A *finite continued fraction* is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_{k-1} + \frac{1}{a_k}}}}}$$

where a_0, a_1, \dots, a_k are all in \mathbb{Z}_0^+ . We'll denote this as $[a_0, a_1, \dots, a_k]$.

Problem 8:

Write each of the following as a continued fraction.

Hint: Solve for one a_n at a time.

- $5/12$
- $5/3$
- $33/23$
- $37/31$

Problem 9:

Write each of the following continued fractions as a regular fraction in lowest terms:

- $[2, 3, 2]$
- $[1, 4, 6, 4]$
- $[2, 3, 2, 3]$
- $[9, 12, 21, 2]$

Problem 10:

Let $\frac{p}{q}$ be a positive rational number in lowest terms. Perform the Euclidean algorithm to obtain the following sequence:

$$\begin{aligned} p &= q_0q + r_1 \\ q &= q_1r_1 + r_2 \\ r_1 &= q_2r_2 + r_3 \\ &\vdots \\ r_{k-1} &= q_kr_k + 1 \\ r_k &= q_{k+1} \end{aligned}$$

We know that we will eventually get 1 as the remainder because p and q are relatively prime. Show that $p/q = [q_0, q_1, \dots, q_{k+1}]$.

Problem 11:

Repeat Problem 8 using the method outlined in Problem 10.

Definition 12:

An *infinite continued fraction* is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

where a_0, a_1, a_2, \dots are in \mathbb{Z}_0^+ . To prove that this expression actually makes sense and equals a finite number is beyond the scope of this worksheet, so we assume it for now. This is denoted $[a_0, a_1, a_2, \dots]$.

Problem 13:

Using a calculator, compute the first five terms of the continued fraction expansion of the following numbers. Do you see any patterns?

- $\sqrt{2}$
- $\pi \approx 3.14159\dots$
- $\sqrt{5}$
- $e \approx 2.71828\dots$

Problem 14:

Show that an $\alpha \in \mathbb{R}^+$ can be written as a finite continued fraction if and only if α is rational.

Hint: For one of the directions, use Problem 10

Definition 15:

The continued fraction $[a_0, a_1, a_2, \dots]$ is *periodic* if it ends in a repeating sequence of digits. A few examples are below. We denote the repeating sequence with a line.

- $[1, 2, 2, 2, \dots] = [1, \overline{2}]$ is periodic.
- $[1, 2, 3, 4, 5, \dots]$ is not periodic.
- $[1, 3, 7, 6, 4, 3, 4, 3, 4, 3, \dots] = [1, 3, 7, 6, \overline{4, 3}]$ is periodic.
- $[1, 2, 4, 8, 16, \dots]$ is not periodic.

Problem 16:

- Show that $\sqrt{2} = [1, \overline{2}]$.
- Show that $\sqrt{5} = [1, \overline{4}]$.

Hint: use the same strategy as Problem 13 but without a calculator.

Problem 17: Challenge I

Express the following continued fractions in the form $\frac{a+\sqrt{b}}{c}$ where a , b , and c are integers:

- $[\overline{1}]$
- $[\overline{2, 5}]$
- $[1, 3, \overline{2, 3}]$

Problem 18: Challenge II

Let $\alpha = [a_0, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+p}}]$ be any periodic continued fraction.

Prove that α is of the form $\frac{a+\sqrt{b}}{c}$ for some integers a, b, c where b is not a perfect square.

Problem 19: Challenge III

Prove that any number of the form $\frac{a+\sqrt{b}}{c}$ where a, b, c are integers and b is not a perfect square can be written as a periodic continued fraction.

Part 3: Convergents

Definition 20:

Let $\alpha = [a_0, a_1, a_2, \dots]$ be an infinite continued fraction (aka an irrational number).

The n th convergent to α is the rational number $[a_0, a_1, \dots, a_n]$ and is denoted $C_n(\alpha)$.

Problem 21:

Calculate the following convergents and write them in lowest terms:

- $C_3([1, 2, 3, 4, \dots])$
- $C_4([0, \overline{2, 3}])$
- $C_5([1, \overline{5}])$

Problem 22:

Recall from last week that $\sqrt{5} = [2, \overline{4}]$. Calculate the first five convergents to $\sqrt{5}$ and write them in lowest terms. Do you notice any patterns?

Hint: Look at the numbers $\sqrt{5} - C_j(\sqrt{5})$ for $1 \leq j \leq 5$

Properties of Convergents

In this section, we want to show that the n th convergent to a real number α is the best approximation of α with the given denominator. Let $\alpha = [a_0, a_1, \dots]$ be fixed, and we will write C_n instead of $C_n(\alpha)$ for short. Let p_n/q_n be the expression of C_n as a rational number in lowest terms. We will eventually prove that $|\alpha - C_n| < \frac{1}{q_n^2}$, and there is no better rational estimate of α with denominator less than or equal to q_n .

First we want the recursive formulas $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ given $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.

Problem 23:

Verify the recursive formula for $1 \leq j \leq 3$ for the convergents C_j of:

- $[1, 2, 3, 4, \dots]$
- $[0, \overline{2, 3}]$
- $[\overline{1, 5}]$

Problem 24: Challenge IV

Prove that $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ by induction.

- As the base case, verify the recursive formulas for $n = 1$ and $n = 2$.
- Assume the recursive formulas hold for $n \leq m$ and show the formulas hold for $m + 1$.

Problem 25:

Using the recursive formula from Problem 24, we will show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

- What is $p_1 q_0 - p_0 q_1$?
- Substitute $a_n p_{n-1} + p_{n-2}$ for p_n and $a_n q_{n-1} + q_{n-2}$ for q_n in $p_n q_{n-1} - p_{n-1} q_n$. Simplify the expression.
- What happens when $n = 2$? Explain why $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

Problem 26: Challenge VI

Similarly derive the formula $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$.

Problem 27:

Recall $C_n = p_n/q_n$. Show that $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1}q_n}$ and $C_n - C_{n-2} = \frac{(-1)^{n-2}a_n}{q_{n-2}q_n}$.

Hint: Use Problem 25 and $p_nq_{n-2} - p_{n-2}q_n = (-1)^{n-2}a_n$ respectively

In Problem 22, the value $\alpha - C_n$ alternated between negative and positive and $|\alpha - C_n|$ got smaller each step. Using the relations in Problem 27, we can prove that this is always the case. Specifically, α is always between C_n and C_{n+1} .

Problem 28:

Let's figure out how well the n th convergents estimate α . We will show that $|\alpha - C_n| < \frac{1}{q_n^2}$.

- Note that $|C_{n+1} - C_n| = \frac{1}{q_nq_{n+1}}$.
- Why is $|\alpha - C_n| \leq |C_{n+1} - C_n|$?
- Conclude that $|\alpha - C_n| < \frac{1}{q_n^2}$.

We are now ready to prove a fundamental result in the theory of rational approximation.

Problem 29: Dirichlet's approximation theorem

Let α be any irrational number. Prove that there are infinitely many rational numbers $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.

Problem 30: Challenge VII

Prove that if α is *rational*, then there are only *finitely* many rational numbers $\frac{p}{q}$ satisfying

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

The above result shows that the n th convergents estimate α extremely well. Are there better estimates for α if we want small denominators? In order to answer this question, we introduce the Farey sequence.

Definition 31:

The *Farey sequence* of order n is the set of rational numbers between 0 and 1 whose denominators (in lowest terms) are $\leq n$, arranged in increasing order.

Problem 32:

List the Farey sequence of order 4. Now figure out the Farey sequence of order 5 by including the relevant rational numbers in the Farey sequence of order 4.

Problem 33:

Let $\frac{a}{b}$ and $\frac{c}{d}$ be consecutive elements of the Farey sequence of order 5. What does $bc - ad$ equal?

Problem 34: Challenge VIII

Prove that $bc - ad = 1$ for $\frac{a}{b}$ and $\frac{c}{d}$ consecutive rational numbers in Farey sequence of order n .

- In the plane, draw the triangle with vertices $(0,0)$, (b,a) , (d,c) . Show that the area A of this triangle is $\frac{1}{2}$ using Pick's Theorem. Recall that Pick's Theorem states $A = \frac{B}{2} + I - 1$ where B is the number of lattice points on the boundary and I is the number of points in the interior.

Hint: $B=3$ and $I=0$

- Show that the area of the triangle is also given by $\frac{1}{2}|ad - bc|$.
- Why is $bc > ad$?
- Conclude that $bc - ad = 1$.

Problem 35:

Use the result of Problem 34 to show that there is no rational number between C_{n-1} and C_n with denominator less than or equal to q_n . Conclude that if a/b is any rational number with $b \leq q_n$, then $|\alpha - \frac{a}{b}| \geq |\alpha - \frac{p_n}{q_n}|$

Problem 36: Challenge X

Prove the following strengthening of Dirichlet's approximation theorem. If α is irrational, then there are infinitely many rational numbers $\frac{p}{q}$ satisfying $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$.

- Prove that $(x + y)^2 \geq 4xy$ for any real x, y .
- Let p_n/q_n be the n th convergent to α . Prove that

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right|^2 \geq 4 \left| \frac{p_n}{q_n} - \alpha \right| \left| \frac{p_{n+1}}{q_{n+1}} - \alpha \right|$$

Hint: α lies in between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$

- Prove that either $\frac{p_n}{q_n}$ or $\frac{p_{n+1}}{q_{n+1}}$ satisfies the desired inequality (Hint: proof by contradiction).
- Conclude that there are infinitely many rational numbers $\frac{p}{q}$ satisfying $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$.