To Mock a Mockingbird

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Based on a book of the same name.

Instructor's Handout

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Part 1: Introduction

A certain enchanted forest is inhabited by talking birds. Each of these birds has a name, and will respond whenever it hears the name of another. Suppose you are exploring this forest and come across the bird A. You call the name of bird B. A hears you and responds with the name of some other bird, which we will designate AB.
Bird AB is, by definition, A's response to B.

As you wander around this forest, you quickly discover two interesting facts:

A: A's response to B mustn't be the same as B's response to A.

B: Given three birds A, B, and C, $(AB)C$ and $A(BC)$ are not necessarily the same bird.
Bird $A(BC)$ is A's response to bird BC, while $(AB)C$ is AB's response to C.
Thus, ABC is ambiguous. Parenthesis are mandatory.

You also find that this forest has two laws:

A: *The Law of Composition*:
For any two birds A and B, there must be a bird C so that $Cx = A(Bx)$

B: *The Law of the Mockingbird*:
The forest must contain the Mockingbird M, which always satisfies $Mx = xx$.
In other words, the Mockingbird's response to any bird x is the same as x's response to itself.

Definition 1:
We say a bird A is fond of a bird B if A responds to B with B.
In other words, A is fond of B if $AB = B$.

Definition 2:
We say a bird C *composes* A with B if for any bird x,

$$Cx = A(Bx)$$

In other words, this means that C's response to x is the same as A's response to B's response to x.
Note that C is exactly the kind of bird L_1 guarantees.

Part 2: To Mock a Mockingbird

Problem 3:
Mark tells you that any bird A is fond of at least one other bird.
Complete his proof.

```
let A           # Let A be any any bird.
let Cx = A(Mx)  # Define C as the composition of A and M

# The rest is up to you.
CC = ??
```

Things you will need:

`Law:` There exists a Mockingbird, $Mx := xx$
`Def:` A is fond of B if $AB = B$

Solution

```
01 let A           # Let A be any any bird.
02 let Cx = A(Mx)  # Define C as the composition of A and M
03 CC = A(MC)
04    = A(CC)  ■
```

Problem 4:
We say a bird A is *egocentric* if it is fond of itself.
Show that the laws of the forest guarantee that at least one bird is egocentric.

Things you will need:

`Law:` There exists a Mockingbird, $Mx := xx$
`Def:` A is fond of B if $AB = B$
`Lem:` Any bird is fond of at least one bird.

Solution

```
01 # We know M is fond of at least one bird.
02 let E so that ME = E
03
04 ME = E     # By definition of fondness
05 ME = EE    # By definition of M
06 ⇒ EE = E  ■
```

Definition 5:
We say a bird A is *agreeable* if for all birds B, there is at least one bird x on which A and B agree. In other words, A is agreeable if given any B, we can find a bird x satisfying $Ax = Bx$.

Problem 6:
Is the Mockingbird agreeable?

Solution

We know that $Mx = xx$.
From this definition, we see that M agrees with any x on x itself.

Problem 7:
Take two birds A and B. Let C be their composition.
Show that if C is agreeable, A is agreeable.

```
# Given information
let A, B
let Cx = A(Bx)

let D            # Arbitrary bird
let Ex = D(Bx)   # Define E as the composition of D and B
Cy = ??
```

Things you will need:

`Def:` A is agreeable if $Ax = Bx$ for all B with some x.
`Law:` For any A, B, there is C defined by Cx = A(Bx)

Solution

```
01 # Given information
02 let A, B
03 let Cx = A(Bx)
04
05 let D                  # Arbitrary bird
06 let Ex = D(Bx)         # Define E as the composition of D and B
07 let y so that Cy = Ey  # Such a y must exist because C is agreeable
08
09 A(By) = Ey
10       = D(By)  ■
```

Problem 8:
Given three arbitrary birds A, B, and C, show that there exists a bird D satisfying $Dx = A(B(Cx))$

Solution

```
01 let A, B, C
02
03 # Invoke the Law of Composition:
04 let Qx = B(Cx)
05 let Dx = A(Qx)
06
07 Dx = A(Qx)
08   = A(B(Cx))  ■
```

Definition 9:
We say two birds A and B are *compatible* if there are birds x and y so that $Ax = y$ and $By = x$. Note that x and y may be the same bird.

Problem 10:
Show that any two birds in this forest are compatible.

```
let A, B
let Cx = A(Bx)
```

Things you will need:

`Law:` Law of composition
`Lem:` Any bird is fond of at least one bird.

Solution

```
01 let A, B
02
03 let Cx = A(Bx)  # Composition
04 let y = Cy      # Let C be fond of y
05
06 Cy = y
07    = A(By)
08
09 let x = By  # Rename By to x
10 Ax = y  ■
```

Problem 11:
Show that any bird that is fond of at least one bird is compatible with itself.

Solution

```
01 let A
02 let x so that Ax = x  # A is fond of at least one other bird
03 Ax = x  ■
```

Solution (continued)

That's it.

Part 3: The Curious Kestrel

Definition 12:
Recall that a bird is *egocentric* if it is fond of itself.
A bird is *hopelessly egocentric* if $Bx = B$ for all birds x.

Definition 13:
More generally, we say that a bird A is *fixated* on a bird B if $Ax = B$ for all x.
Convince yourself that a hopelessly egocentric bird is fixated on itself.

Problem 14:
Say A is fixated on B. Is A fond of B?

Solution

Yes! See the following proof.

```
01 let A
02 let B so that Ax = B
03 ⇒ AB = B ■
```

Definition 15:
The *Kestrel* K is defined by the following relationship:

$$(Kx)y = x \quad \forall x, y$$

In other words, this means that for every bird x, the bird Kx is fixated on x.

Problem 16:
Show that an egocentric Kestrel is hopelessly egocentric.

Solution

```
01 KK = K
02 ⇒ (KK)y = K  # By definition of the Kestrel
03 ⇒ Ky = K ■  # By 01
```

Problem 17:
Assume the forest contains a Kestrel.
Given the Law of Composition and the Law of the Mockingbird, show that at least one bird is hopelessly egocentric.

Things you will need:

`Def:` K is defined by $(Kx)y = x$
`Def:` A is fond of B if $AB = B$
`???:` You'll need one more result from the previous section. Good luck!

Solution

The final piece is a lemma we proved earlier:
Any bird is fond of at least one bird

```
01 let A so that KA = A   # Any bird is fond of at least one bird
02 (KA)y = y              # By definition of the kestrel
03 ⇒ Ay = A ■  # By 01
```

Problem 18: Kestrel Left-Cancellation
In general, $Ax = Ay$ does not imply $x = y$. However, this is true if A is K.
Show that $Kx = Ky \implies x = y$.

```
# This is a hint.
let x, y so that Kx = Ky
```

Solution

```
01 let x, y so that Kx = Ky
02 let z
03
04 (Kx)z = (Ky)z  # By 01
05
06 # By the definition of K
07 (Kx)z = x
08 (Ky)z = y
09
10 ⇒ x = (Kx)z = (Ky)z = y ■
```

Problem 19:
Show that if K is fond of Kx, K is fond of x.

Solution

```
01 let x so that K(Kx) = Kx
02 (K(Kx))y = (Kx)y
03          = Kx  # By definition of K
04 x = Kx  # By 03 and definition of K
```

Problem 20:
An egocentric Kestrel must be extremely lonely. Why is this?

Solution

If a Kestrel is egocentric, it must be the only bird in the forest!

```
01 # Given
02 Kx = K for some x
03 # We have shown that an egocentric kestrel is hopelessly egocentric
04 Kx = K for all x
05
06 let x, y
07 Kx = K
08 Ky = K
09 Kx = Ky
10 x = y for all x, y  # By Problem 18
11 x = y = K ■  # By 10, and since K exists
```