

# De Bruijn Sequences

Prepared by Mark on January 23, 2025  
Based on a handout by Glenn Sun

---

## Part 1: Introduction

**Example 1:**

A certain electronic lock has two buttons: 0 and 1. It opens as soon as the correct two-digit code is entered, completely ignoring previous inputs. For example, if the correct code is 10, the lock will open once the sequence 010 is entered.

Naturally, there are  $2^2 = 4$  possible combinations that open this lock.

If we don't know the lock's combination, we could try to guess it by trying all four combinations. This would require eight key presses: 0001101100.

**Problem 2:**

There is, of course, a better way.

Unlock this lock with only 5 keypresses.

Now, consider the same lock, now set with a three-digit binary code.

**Problem 3:**

How many codes are possible?

**Problem 4:**

Show that there is no solution with fewer than three keypresses

**Problem 5:**

What is the shortest sequence that is guaranteed to unlock the lock?

*Hint:* You'll need 10 digits.

## Part 2: Words

**Definition 6:**

An *alphabet* is a set of symbols.

For example,  $\{0, 1\}$  is an alphabet of two symbols, and  $\{a, b, c\}$  is an alphabet of three.

**Definition 7:**

A *word* over an alphabet  $A$  is a sequence of symbols in that alphabet.

For example, 00110 is a word over the alphabet  $\{0, 1\}$ .

We'll let  $\emptyset$  denote the empty word, which is a valid word over any alphabet.

**Definition 8:**

Let  $v$  and  $w$  be words over the same alphabet.

We say  $v$  is a *subword* of  $w$  if  $v$  is contained in  $w$ .

In other words,  $v$  is a subword of  $w$  if we can construct  $v$  by removing a few characters from the start and end of  $w$ .

For example, 11 is a subword of 011, but 00 is not.

**Definition 9:**

Recall ???. Let's generalize this to the *n-subword problem*:

Given an alphabet  $A$  and a positive integer  $n$ , we want a word over  $A$  that contains all possible length- $n$  subwords. The shortest word that solves a given  $n$ -subword problem is called the *optimal solution*.

**Problem 10:**

List all subwords of 110.

*Hint:* There are six.

**Definition 11:**

Let  $\mathcal{S}_n(w)$  be the number of subwords of length  $n$  in a word  $w$ .

**Problem 12:**

Find the following:

- $\mathcal{S}_n(101001)$  for  $n \in \{0, 1, \dots, 6\}$
- $\mathcal{S}_n(\text{abccac})$  for  $n \in \{0, 1, \dots, 6\}$

**Problem 13:**

Let  $w$  be a word over an alphabet of size  $k$ .

Prove the following:

- $\mathcal{S}_n(w) \leq k^n$
- $\mathcal{S}_n(w) \geq \mathcal{S}_{n-1}(w) - 1$
- $\mathcal{S}_n(w) \leq k \times \mathcal{S}_{n-1}(w)$

**Definition 14:**

Let  $v$  and  $w$  be words over the same alphabet.

The word  $vw$  is the word formed by writing  $v$  after  $w$ .

For example, if  $v = 1001$  and  $w = 10$ ,  $vw$  is  $100110$ .

**Problem 15:**

Let  $F_k$  denote the word over the alphabet  $\{0, 1\}$  obtained from the following relation:

$$F_0 = 0; \quad F_1 = 1; \quad F_k = F_{k-1}F_{k-2}$$

We'll call this the *Fibonacci word* of order  $k$ .

- What are  $F_3$ ,  $F_4$ , and  $F_5$ ?
- Compute  $S_0$  through  $S_5$  for  $F_5$ .
- Show that the length of  $F_k$  is the  $(k + 2)^{\text{th}}$  Fibonacci number.

*Hint:* Induction.

**Problem 16:**

Let  $C_k$  denote the word over the alphabet  $\{0, 1\}$  obtained by concatenating the binary representations of the integers  $0, \dots, 2^k - 1$ . For example,  $C_1 = 01$ ,  $C_2 = 011011$ , and  $C_3 = 011011100101110111$ .

- Compute  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ ,  $\mathcal{S}_2$ , and  $\mathcal{S}_3$  for  $C_3$ .
- Show that  $\mathcal{S}_k(C_k) = 2^k - 1$ .
- Show that  $\mathcal{S}_n(C_k) = 2^n$  for  $n < k$ .

*Hint:* If  $v$  is a subword of  $w$  and  $w$  is a subword of  $u$ ,  $v$  must be a subword of  $u$ . In other words, the “subword” relation is transitive.

**Problem 17:**

Convince yourself that  $C_{n+1}$  provides a solution to the  $n$ -subword problem over  $\{0, 1\}$ .

*Note:*  $C_{n+1}$  may or may not be an *optimal* solution—but it is a *valid* solution. Which part of ?? shows that this is true?

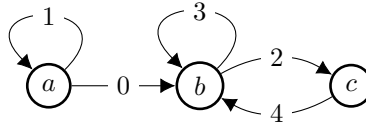
## Part 3: De Bruijn Words

Before we continue, we'll need to review some basic graph theory.

### Definition 18:

A *directed graph* consists of nodes and directed edges.

An example is shown below. It consists of three vertices (labeled  $a, b, c$ ), and five edges (labeled  $0, \dots, 4$ ).



### Definition 19:

A *path* in a graph is a sequence of adjacent edges,

In a directed graph, edges  $a$  and  $b$  are adjacent if  $a$  ends at the node which  $b$  starts at.

For example, consider the graph above.

The edges 1 and 0 are adjacent, since you can take edge 0 after taking edge 1.

0 starts where 1 ends.

0 and 1, however, are not: 1 does not start at the edge at which 0 ends.

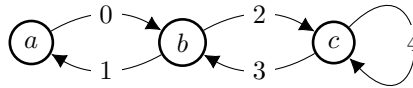
### Definition 20:

An *Eulerian path* is a path that visits each edge of a graph exactly once.

An *Eulerian cycle* is an Eulerian path that starts and ends on the same node.

### Problem 21:

Find the single unique Eulerian cycle in the graph below.



### Theorem 22:

A directed graph contains an Eulerian cycle iff...

- There is a path between every pair of nodes, and
- every node has as many “in” edges as it has “out” edges.

If a graph contains an Eulerian cycle, it must contain an Eulerian path. (why?)

Some graphs contain an Eulerian path, but not a cycle. In this case, both conditions above must still hold, but the following exceptions are allowed:

- There may be at most one node where (number in – number out) = 1
- There may be at most one node where (number in – number out) = –1

*Note:* Either both exceptions occur, or neither occurs. Bonus problem: why?

We won't provide a proof of this theorem today. However, you should convince yourself that it is true: if any of these conditions are violated, why do we know that an Eulerian cycle (or path) cannot exist?

**Definition 23:**

Now, consider the  $n$ -subword problem over  $\{0, 1\}$ .

We'll call the optimal solution to this problem a *De Bruijn*<sup>1</sup> word of order  $n$ .

**Problem 24:**

Let  $w$  be the an order- $n$  De Bruijn word, and denote its length with  $|w|$ .

Show that the following bounds always hold:

- $|w| \leq n2^n$
- $|w| \geq 2^n + n - 1$

**Remark 25:**

Now, we'd like to show that the length of a De Bruijn word is always  $2^n + n - 1$

That is, that the optimal solution to the subword problem always has  $2^n + n - 1$  letters.

We'll do this by construction: for a given  $n$ , we want to build a word with length  $2^n + n - 1$  that solves the binary  $n$ -subword problem.

**Definition 26:**

Consider a  $n$ -length word  $w$ .

The *prefix* of  $w$  is the word formed by the first  $n - 1$  letters of  $w$ .

The *suffix* of  $w$  is the word formed by the last  $n - 1$  letters of  $w$ .

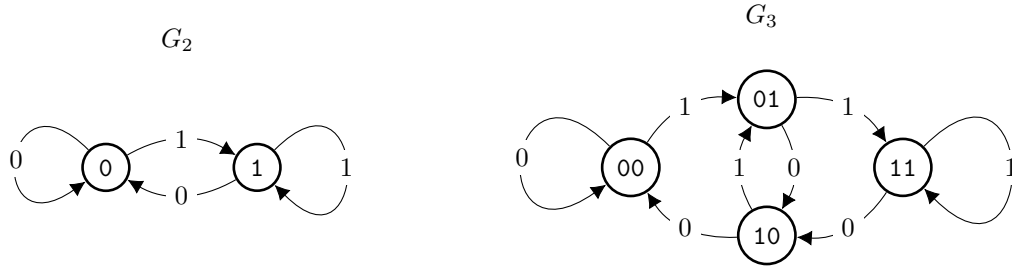
For example, the prefix of the word 1101 is 110, and its suffix is 101. The prefix and suffix of any one-letter word are both  $\emptyset$ .

**Definition 27:**

A *De Bruijn graph* of order  $n$ , denoted  $G_n$ , is constructed as follows:

- Nodes are created for each word of length  $n - 1$ .
- A directed edge is drawn from  $a$  to  $b$  if the suffix of  $a$  matches the prefix of  $b$ .  
Note that a node may have an edge to itself.
- We label each edge with the last letter of  $b$ .

$G_2$  and  $G_3$  are shown below.




---

<sup>1</sup>Dutch. Rhymes with “De Grown.”

**Problem 28:**  
Draw  $G_4$ .

**Problem 29:**

- Show that  $G_n$  has  $2^{n-1}$  nodes and  $2^n$  edges;
- that each node has two outgoing edges;
- and that there are as many edges labeled 0 as are labeled 1.

**Problem 30:**

Show that  $G_4$  always contains an Eulerian path.

*Hint: ??*

**Theorem 31:**

We can now easily construct De Bruijn words for a given  $n$ :

- Construct  $G_n$ ,
- find an Eulerian cycle in  $G_n$ ,
- then, construct a De Bruijn word by writing the label of our starting vertex, then appending the label of every edge we travel.

**Problem 32:**

Find De Bruijn words of orders 2, 3, and 4.

Let's quickly show that the process described in ?? indeed produces a valid De Bruijn word.

**Problem 33:**

How long will a word generated by the above process be?

**Problem 34:**

Show that a word generated by the process in ?? contains every possible length- $n$  subword. In other words, show that  $\mathcal{S}_n(w) = 2^n$  for a generated word  $w$ .

**Remark 35:**

- We found that ?? generates a word with length  $2^n + n - 1$  in ??,
- and we showed that this word always solves the  $n$ -subword problem in ??.
- From ??, we know that any solution to the binary  $n$ -subword problem must have at least  $2^n + n - 1$  letters.
- Finally, ?? guarantees that it is possible to generate such a word in any  $G_n$ .

Thus, we have shown that the process in ?? generates ideal solutions to the  $n$ -subword problem, and that such solutions always exist. We can now conclude that for any  $n$ , the binary  $n$ -subword problem may be solved with a word of length  $2^n + n - 1$ .

## Part 4: Line Graphs

### Problem 36:

Given a graph  $G$ , we can construct a graph called the *line graph* of  $G$  (denoted  $\mathcal{L}(G)$ ) by doing the following:

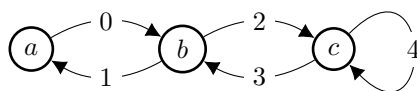
- Creating a node in  $\mathcal{L}(G)$  for each edge in  $G$
- Drawing a directed edge between every pair of nodes  $a, b$  in  $\mathcal{L}(G)$  if the corresponding edges in  $G$  are adjacent.

That is, if edge  $b$  in  $G$  starts at the node at which  $a$  ends.

### Problem 37:

Draw the line graph for the graph below.

Have an instructor check your solution.



### Definition 38:

We say a graph  $G$  is *connected* if there is a path between any two vertices of  $G$ .

### Problem 39:

Show that if  $G$  is connected,  $\mathcal{L}(G)$  is connected.

**Definition 40:**

Consider  $\mathcal{L}(G_n)$ , where  $G_n$  is the  $n^{\text{th}}$  order De Bruijn graph.

We'll need to label the vertices of  $\mathcal{L}(G_n)$ . To do this, do the following:

- Let  $a$  and  $b$  be nodes in  $G_n$
- Let  $x$  be the first letter of  $a$
- Let  $y$ , the last letter of  $b$
- Let  $\bar{p}$  be the prefix/suffix that  $a$  and  $b$  share.

Note that  $a = x\bar{p}$  and  $b = \bar{p}y$ ,

Now, relabel the edge from  $a$  to  $b$  as  $x\bar{p}y$ .

Use these new labels to name nodes in  $\mathcal{L}(G_n)$ .

**Problem 41:**

Construct  $\mathcal{L}(G_2)$  and  $\mathcal{L}(G_3)$ . What do you notice?

*Hint:* What are  $\mathcal{L}(G_2)$  and  $\mathcal{L}(G_3)$ ? We've seen them before!

You may need to re-label a few edges.

## Part 5: Sturmian Words

A De Bruijn word is the shortest word that contains all subwords of a given length.

Let's now solve a similar problem: given an alphabet, we want to construct a word that contains exactly  $m$  distinct subwords of length  $n$ .

In general, this is a difficult problem. We'll restrict ourselves to a special case:

We'd like to find a word that contains exactly  $m + 1$  distinct subwords of length  $m$  for all  $m < n$ .

### Definition 42:

We say a word  $w$  is a *Sturmian word* of order  $n$  if  $\mathcal{S}_m(w) = m + 1$  for all  $m \leq n$ .

We say  $w$  is a *minimal* Sturmian word if there is no shorter Sturmian word of that order.

### Problem 43:

Show that the length of a Sturmian word of order  $n$  is at least  $2n$ .

**Problem 44:**

Construct  $R_3$  by removing four edges from  $G_3$ .

Show that each of the following is possible:

- $R_3$  does not contain an Eulerian path.
- $R_3$  contains an Eulerian path, and this path constructs a word  $w$  with  $\mathcal{S}_3(w) = 4$  and  $\mathcal{S}_2(w) = 4$ .
- $R_3$  contains an Eulerian path, and this path constructs a word  $w$  that is a minimal Sturmian word of order 3.

**Problem 45:**

Construct  $R_2$  by removing one edge from  $G_2$ , then construct  $\mathcal{L}(R_2)$ .

- If this line graph has four edges, set  $R_3 = \mathcal{L}(R_2)$ .
- If not, remove one edge from  $\mathcal{L}(R_2)$  so that an Eulerian path still exists and set  $R_3$  to the resulting graph.

Label each edge in  $R_3$  with the last letter of its target node.

Let  $w$  be the word generated by an Eulerian path in this graph, as before.

Attempt the above construction a few times. Is  $w$  a minimal Sturmian word?

**Theorem 46:**

We can construct a minimal Sturmian word of order  $n \geq 3$  as follows:

- Start with  $G_2$ , create  $R_2$  by removing one edge.
- Construct  $\mathcal{L}(G_2)$ , remove an edge if necessary.  
 The resulting graph must have an 4 edges and an Eulerian path. Call this  $R_3$ .
- Repeat the previous step to construct a sequence of graphs  $R_n$ .  
 $R_{n-1}$  is used to create  $R_n$ , which has  $n + 1$  edges and an Eulerian path.  
 Label edges with the last letter of their target vertex.
- Construct a word  $w$  using the Eulerian path, as before.  
 This is a minimal Sturmian word.

For now, assume this theorem holds. We'll prove it in the next few problems.

**Problem 47:**

Construct a minimal Sturmian word of order 4.

**Problem 48:**

Construct a minimal Sturmian word of order 5.

**Problem 49:**

Argue that the words we get by ?? are minimal Sturmian words.

That is, the word  $w$  has length  $2n$  and  $\mathcal{S}_m(w) = m + 1$  for all  $m \leq n$ .