
Random Walks and Resistance

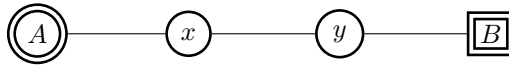
Prepared by Mark on January 23, 2025
Based on a handout by Aaron Anderson

Part 1: Random Walks

Consider the graph below. A particle sits on some node n . Every second, this particle moves left or right with equal probability. Once it reaches node A or B , it stops.

We would like to compute the probability of our particle stopping at node A .

In other words, we want a function $P : \text{Nodes} \rightarrow [0, 1]$ that maps each node of the graph to the probability that our particle stops at A .



Problem 1:

What are $P(A)$ and $P(B)$ in the graph above?

Note that these values hold for all graphs.

Problem 2:

Find an expression for $P(x)$ in terms of $P(y)$ and $P(A)$.

Find an expression for $P(y)$ in terms of $P(x)$ and $P(B)$.

Problem 3:

Use the previous problems to find $P(x)$ and $P(y)$.

Problem 4:

Say we have a graph G and a particle on node x with neighbors v_1, v_2, \dots, v_n .

Assume that our particle is equally likely to travel to each neighbor.

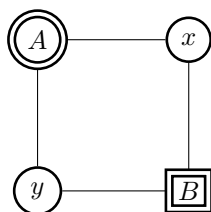
Find $P(x)$ in terms of $P(v_1), P(v_2), \dots, P(v_n)$.

Problem 5:

In general, how do we find $P(n)$ for any node n ?

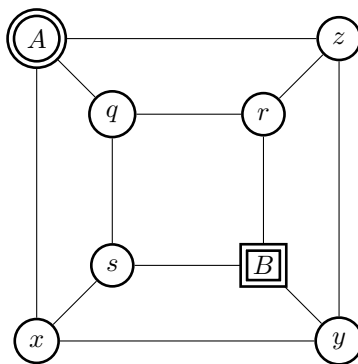
Problem 6:

Find $P(n)$ for all nodes in the graph below.

**Problem 7:**

Find $P(n)$ for all nodes in the graph below.

Note that this is the graph of a cube with A and B on opposing vertices.

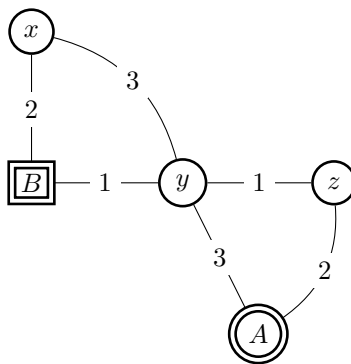


Definition 8:

Let us now take a look at weighted graphs. The problem remains the same: we want to compute the probability that our particle stops at node A , but our graphs will now feature weighted edges. The probability of our particle taking a certain edge is proportional to that edge's weight.

For example, if our particle is on node y of the graph below, it has a $\frac{3}{8}$ probability of moving to x and a $\frac{1}{8}$ probability of moving to z .

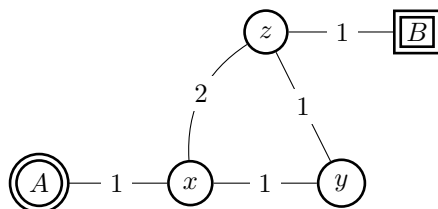
Note that $3 + 3 + 1 + 1 = 8$.

**Problem 9:**

Say a particle on node x has neighbors v_1, v_2, \dots, v_n with weights w_1, w_2, \dots, w_n . The edge (x, v_1) has weight w_1 . Find $P(x)$ in terms of $P(v_1), P(v_2), \dots, P(v_n)$.

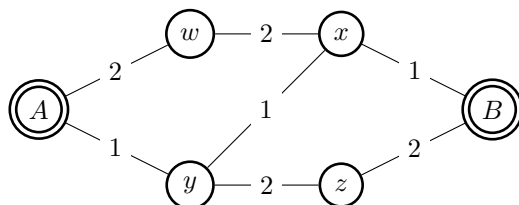
Problem 10:

Consider the following graph. Find $P(x)$, $P(y)$, and $P(z)$.

**Problem 11:**

Consider the following graph.

What expressions can you find for $P(w)$, $P(x)$, $P(y)$, and $P(z)$?



Solve this system of equations.

Hint: Use symmetry. $P(w) = 1 - P(z)$ and $P(x) = 1 - P(y)$. Why?

Part 2: Circuits

An *electrical circuit* is a graph with a few extra properties, called *current*, *voltage*, and *resistance*. In the definitions below, let X be the set of nodes in a circuit.

- **Voltage** is a function $V : X \rightarrow \mathbb{R}$ that assigns a number to each node of our graph. In any circuit, we pick a “ground” node, and define the voltage¹ there as 0. We also select a “source” node, and define its voltage as 1. Intuitively, you could say we’re connecting the ends of a 1-volt battery to our source and ground nodes.
- **Current** is a function $I : X^2 \rightarrow \mathbb{R}$ that assigns a number to each *oriented edge* in our graph. An “oriented edge” is just an ordered pair of nodes (n_1, n_2) . Current through an edge (a, b) is a measure of the flow of charge from a to b . Naturally, $I(a, b) = -I(b, a)$.
- **Resistance** is a function $R : X^2 \rightarrow \mathbb{R}_0^+$ that represents a certain edge’s resistance to the flow of current through it. Resistance is a property of each *link* between nodes, so order doesn’t matter: $R(a, b) = R(b, a)$.

It is often convenient to compare electrical circuits to systems of pipes. Say we have a pipe from point A to point B . The size of this pipe represents resistance (smaller pipe \implies more resistance), the pressure between A and B is voltage, and the speed water flows through it is to current.

Definition 12: Ohm’s law

With this “pipe” analogy in mind, you may expect that voltage, current, and resistance are related: if we make our pipe bigger (and change no other parameters), we’d expect to see more current. This is indeed the case! Any circuit obeys *Ohm’s law*, stated below:

$$V(a, b) = I(a, b) \times R(a, b)$$

This handout uses two notations for voltage: two-variable $V(a, b)$ and one-variable $V(a)$. The first represents the voltage between points a and b , better reflecting reality (see the footnote below). The second measures the voltage between a and ground, and is more convenient to use in equations. **Try to use the single-variable notation in your equations.** Convince yourself that $V(a, b) = V(a) - V(b)$.

Definition 13: Kirchoff’s law

The second axiom of electrical circuits is also fairly simple. *Kirchoff’s law* states that the sum of all currents connected to a given edge is zero. You can think of this as “conservation of mass”: nodes in our circuit do not create or destroy electrons, they simply pass them around to other nodes. Formally, we can state this as follows:

Let x be a node in our circuit and N_x the set of its neighbors. We then have

$$\sum_{b \in N_x} I(x, b) = 0$$

which must hold at every node **except the source and ground vertices**.

Hint: Keep this exception in mind, it is used in a few problems later on.

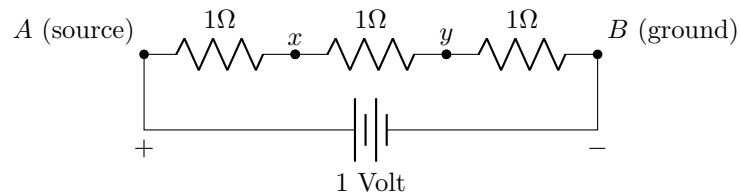
¹In the real world, voltage is always measured *between two points* on a circuit. Voltage is defined as the *difference* in electrical charge between two points. Hence, voltage is a function of two nodes.

Note that this is different than current and resistance, which aren’t functions of two arbitrary nodes — rather, they are functions of *edges* (i.e., two adjacent nodes).

Consider the circuit below. **This the graph from ??**, turned into a circuit by:

- Replacing all edges with 1Ω resistors
- Attaching a 1 volt battery between A and B

Note that the battery between A and B isn't really an edge. It exists only to create a potential difference between the two nodes.



Problem 14:

From the circuit diagram above, we immediately know that $V(A) = 1$ and $V(B) = 0$.

What equations related to the currents out of x and y does Kirchoff's law give us?

Hint: Current into x = current out of x

Problem 15:

Use Ohm's law to turn the equations from ?? into equations about voltage and resistance.

Find an expression for $V(x)$ and $V(y)$ in terms of other voltages, then solve the resulting system of equations. Does your result look familiar?

Part 3: The Equivalence

In the last problem, we found that the equations for $V(x)$ were the same as the equations for $P(x)$ on the same graph. It turns out that this is true in general: problems about voltage in circuits directly correspond to problems about probability in graphs. We'll spend the next section proving this fact.

Definition 16:

For the following problems, *conductance* will be more convenient than resistance.

The definition of conductance is quite simple:

$$C(a, b) = \frac{1}{R(a, b)}$$

Aside: Resistance is usually measured in Ohms, denoted Ω .

A few good-natured physicists came up with the “mho” (denoted \mathfrak{U}) as a unit of conductance, which is equivalent to an inverse Ohm. Unfortunately, NIST discourages the use of Mhos in favor of the equivalent (and less amusing) “Siemens.”

Problem 17:

Let x be a node in a graph.

Let N_x be the set of x 's neighbors, $w(x, y)$ the weight of the edge between nodes x and y , and W_x the sum of the weights of all edges connected to x .

We saw earlier that the probability function P satisfies the following sum:

$$P(x) = \sum_{b \in N_x} \left(P(b) \times \frac{w(x, b)}{W_x} \right)$$

This was never explicitly stated, but is noted in ??.

Use Ohm's and Kirchoff's laws to show that the voltage function V satisfies a similar sum:

$$V(x) = \sum_{b \in N_x} \left(V(b) \times \frac{C(x, b)}{C_x} \right)$$

where $C(x, b)$ is the conductance of edge (x, b) and C_x is the sum of the conductances of all edges connected to x .

Thus, if $w(a, b) = C(a, b)$, P and V satisfy the same system of linear equations. To finish proving that $P = V$, we now need to show that there can only be one solution to this system. We will do this in the next two problems.

Problem 18:

Let q be a solution to the following equations, where $x \neq a, b$.

$$q(x) = \sum_{b \in N_x} \left(q(b) \times \frac{w(x, b)}{W_x} \right)$$

Show that the maximum and minimum of q are $q(a)$ and $q(b)$ (not necessarily in this order).

Problem 19:

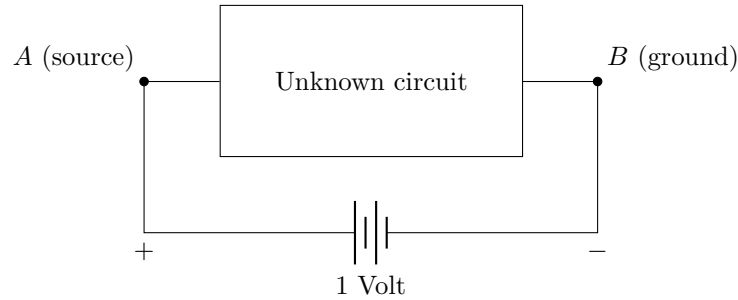
Let p and q be functions that solve our linear system and satisfy $p(A) = q(A) = 1$ and $p(B) = q(B) = 0$.

Show that the function $p - q$ satisfies the equations in ??, and that $p(x) - q(x) = 0$ for every x . Note that $p(x) - q(x) = 0 \ \forall x \implies p = q$

Part 4: Effective Resistance

As we have seen, calculating the properties of a circuit by creating an equation for each vertex is a fairly time-consuming ordeal. Fortunately, there is a better strategy we can use.

Consider a graph (or a circuit) with source and ground vertices. All parts of the circuit that aren't these two vertices are hidden inside a box, as shown below:



What do we know about this box? If this was a physical system, we'd expect that the current flowing out of A is equal to the current flowing into B .

Problem 20:

Using Kirchoff's law, show that the following equality holds.

Remember that we assumed Kirchoff's law holds only at nodes other than A and B .

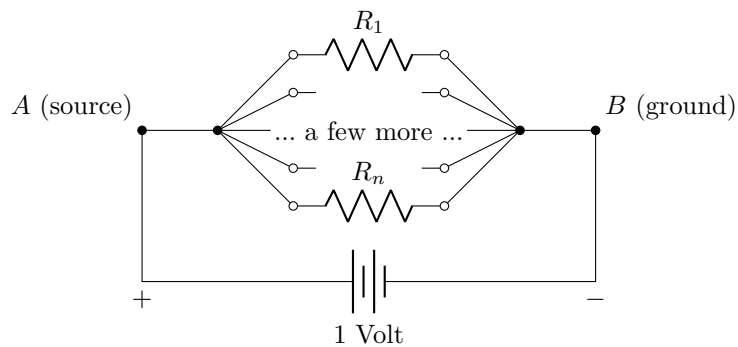
Note: As before, N_x is the set of neighbors of x .

$$\sum_{b \in N_A} I(A, b) = \sum_{b \in N_B} I(b, B)$$

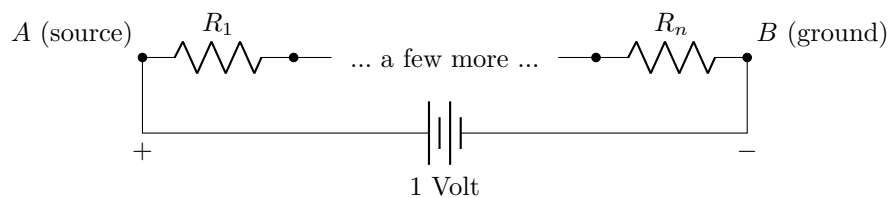
If we call this current $I_A = \sum_{b \in N_A} I(A, b)$, we can pretend that the box contains only one resistor, carrying I_A units of current. Using this information and Ohm's law, we can calculate the *effective resistance* of the box.

Problem 21: Resistors in parallel

Using Ohm's law and Kirchoff's law, calculate the effective resistance R_{eff} of the circuit below.

**Problem 22: Resistors in series**

Using Ohm's law and Kirchoff's law, calculate the effective resistance R_{total} of the circuit below.



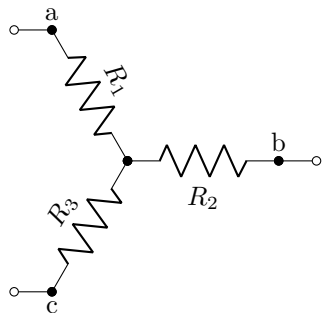
We can now use effective resistance to simplify complicated circuits. Whenever we see the above constructions (resistors in parallel or in series) in a graph, we can replace them with a single resistor of appropriate value.

Problem 23:

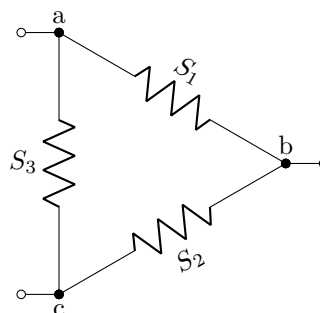
Consider the following circuits. Show that the triangle has the same effective resistance as the star if

- $x = R_1 R_2 + R_1 R_3 + R_2 R_3$
- $S_1 = x/R_3$
- $S_2 = x/R_1$
- $S_3 = x/R_2$

The star:



The triangle:



Problem 24:

Suppose we construct a circuit by connecting the 2^n vertices of an n -dimensional cube with 1Ω resistors. If we place A and B at opposing vertices, what is the effective resistance of this circuit?

Bonus: As $n \rightarrow \infty$, what happens to R_{eff} ?

Note: Leave your answer as a sum.