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# Lattices

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## Instructor's Handout

This file contains solutions and notes.  
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**Definition 1:**

The *integer lattice*  $\mathbb{Z}^n \subset \mathbb{R}^n$  is the set of points with integer coordinates.

**Problem 2:**

Draw  $\mathbb{Z}^2$ .

**Definition 3:**

We say a set of vectors  $\{v_1, v_2, \dots, v_k\}$  *generates*  $\mathbb{Z}^n$  if every lattice point can be written uniquely as

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k$$

for integer coefficients  $a_i$ .

It is fairly easy to show that  $k$  must be at least  $n$ .

**Problem 4:**

Which of the following generate  $\mathbb{Z}^2$ ?

- $\{(1, 2), (2, 1)\}$
- $\{(1, 0), (0, 2)\}$
- $\{(1, 1), (1, 0), (0, 1)\}$

**Solution**

Only the last.

**Problem 5:**

Find a set of two vectors that generates  $\mathbb{Z}^2$ .

Don't say  $\{(0, 1), (1, 0)\}$ , that's too easy.

**Problem 6:**

Find a set of vectors that generates  $\mathbb{Z}^n$ .

**Definition 7:**

A *fundamental region* of a lattice is the parallelepiped spanned by a generating set. The exact shape of this region depends on the generating set we use.

**Problem 8:**

Draw two fundamental regions of  $\mathbb{Z}^2$  using two different generating sets. Verify that their volumes are the same.

## Part 1: Minkowski's Theorem

### Theorem 9: Blichfeldt's Theorem

Let  $X$  be a finite connected region. If the volume of  $X$  is greater than 1,  $X$  must contain two distinct points that differ by an element of  $\mathbb{Z}^n$ . In other words, there exist distinct  $x, y \in X$  so that  $x - y \in \mathbb{Z}^n$ .

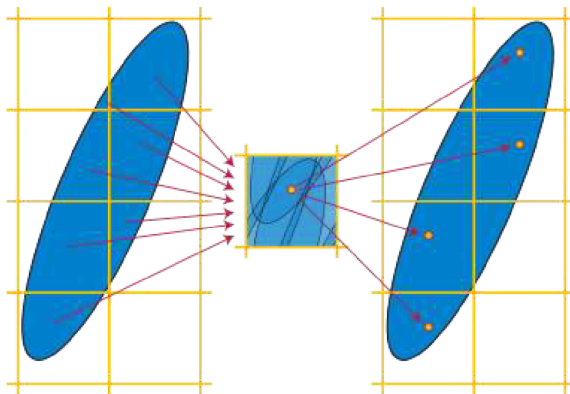
Intuitively, this means that you can translate  $X$  to cover two lattice points at the same time.

### Problem 10:

Draw a region in  $\mathbb{R}^2$  with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint: Area is two-dimensional volume.*

### Problem 11:

The following picture gives an idea for the proof of Blichfeldt's theorem in  $\mathbb{Z}^2$ . Explain the picture and complete the proof.



### Solution

The fundamental region of  $\mathbb{Z}^2$  tiles the plane. Translate these tiles by lattice vectors to stack them on the fundamental region. Then since the union of the intersections of  $X$  with these tiles has area greater 1 and they are stacked on a region of area 1, there must be an overlap by a generalization of the pigeonhole principle (if there were no overlap then the sum of the areas would be less than or equal to 1). Take points  $x, y$  in the overlap. Then  $x - y$  is a lattice point corresponding to the difference in translates, which were lattice points. Hence,  $x - y \in \mathbb{Z}^2$ .

**Problem 12:**

Let  $X$  be a region  $\in \mathbb{R}^2$  of volume  $k$ . How many integral points must  $X$  contain after a translation?

**Solution**

$$\lceil k \rceil$$

**Definition 13:**

A region  $X$  is *convex* if the line segment connecting any two points in  $X$  lies entirely in  $X$ .

**Problem 14:**

- Draw a convex region in the plane.
- Draw a region that is not convex.

**Definition 15:**

We say a region  $X$  is *symmetric* if for all points  $x \in X$ ,  $-x$  is also in  $X$ .

**Problem 16:**

- Draw a symmetric region.
- Draw an asymmetric region.

**Theorem 17: Minkowski's Theorem**

Every convex set in  $\mathbb{R}^n$  that is symmetric with respect to the origin and which has a volume greater than  $2^n$  contains an integral point that isn't zero.

**Problem 18:**

Draw a few sets that satisfy Theorem 17 in  $\mathbb{R}^2$ .

What is the simplest region that has the properties listed above?

**Problem 19:**

Let  $K$  be a region in  $\mathbb{R}^2$  satisfying Theorem 17.

Let  $K'$  be this region scaled by  $\frac{1}{2}$ .

- How does the volume of  $K'$  compare to  $K$ ?
- Show that the sum of any two points in  $K'$  lies in  $K$  *Hint: Use convexity.*
- Apply Blichfeldt's theorem to  $K'$  to prove Minkowski's theorem in  $\mathbb{R}^2$ .

**Problem 20:**

Let  $K$  be a region in  $\mathbb{R}^n$  satisfying Theorem 17. Scale this region by  $\frac{1}{2}$ , called  $K' = \frac{1}{2}K$ .

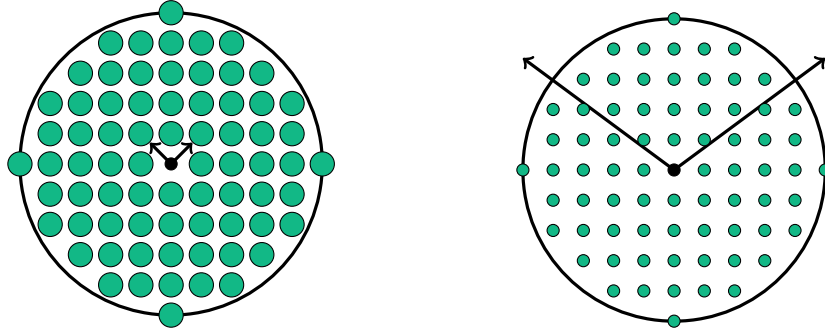
- How does the volume of  $K'$  compare to  $K$ ?
- Show that the sum of any two points in  $K'$  lies in  $K$
- Apply Blichfeldt's theorem to  $K'$  to prove Minkowski's theorem.

**Solution**

- The volume of  $K'$  is  $\frac{1}{2^n}$  the volume of  $K$ .
- Take  $x, y \in K'$ . It follows that  $2x, 2y \in K$ . Since  $K$  is convex, we have that the midpoint of the line segment between  $2x$  and  $2y$  is in  $K$ , and so  $\frac{2x+2y}{2} = x+y \in K$ .
- Since the volume of  $K$  is greater than  $2^n$ , we have the volume of  $K'$  is greater than one. Applying Blichfeldt's theorem, we can find two distinct points  $x, y \in K'$  such that  $x-y \in \mathbb{Z}^n$ . Since  $K'$  is symmetric with respect to the origin, we have that  $-y \in K'$ . Therefore,  $x+(-y) \in K$  by the previous part.  $x \neq y, x-y \neq 0$ , so we have found a nontrivial integer point in  $K$ .

## Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius  $R$ . A tree of radius  $r$  has been planted at every integer point in the circle. If  $r$  is small, you will have a clear line of sight through the orchard. If  $r$  is large, there will be no clear line of sight through in any direction:



### Problem 21:

Show that you will have at least one clear line of sight if  $r < \frac{1}{\sqrt{R^2+1}}$ .

*Hint:* Consider the line segment from  $(0, 0)$  to  $(R, 1)$ . Calculate the distance from the closest integer points to the ray.

### Solution

Consider the ray from the origin to the point  $(R, 1)$ .

The two lattice points closest to this ray are  $(1, 0)$  and  $(R-1, 1)$ . Say the distance from each of these points to the ray is  $\delta$ .

Now, consider the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(R, 1)$ . The area of this triangle is  $\frac{1}{2}$ .

The area of this triangle is also equal to  $\frac{1}{2}\delta\sqrt{R^2+1}$ . By algebra,  $\delta = \frac{1}{\sqrt{R^2+1}}$ .

Therefore, if  $r < \frac{1}{\sqrt{R^2+1}}$ , we will have a clear line of sight given by this ray.

**Problem 22:**

Show that there is no line of sight through the orchard if  $r > \frac{1}{R}$ . You may want to use the following steps:

- Show that there is no line of sight if  $r \geq 1$ .
- Suppose  $r < 1$  and  $r > \frac{1}{R}$ . Then,  $R \geq 2$ . Choose a potential line of sight passing through an arbitrary point  $P$  on the circle. Thicken this line of sight equally on both sides into a rectangle of width  $2r$  tangent to  $P$  and  $-P$ . From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

**Solution**

Suppose  $r < 1$  and let  $L$  be a potential line of sight. Consider the rectangle of width  $2r$  tangent to  $P$  and  $-P$ . Then this is convex and symmetric with respect to the origin. Its area is  $(2R)(2r) > 4\frac{R}{R} = 4$ . By Minkowski, we have a nonzero integral point in this rectangle. Suppose first that the integer point is within the orchard. Then this means that there is a tree whose distance to the line is at most  $r$ . Therefore, this tree blocks the line of sight. Now notice that there is a part of this rectangle that sits outside the orchard. Can the integer point be in this region? This would mean its distance to the origin,  $D$ , would satisfy  $D > R$ . Now since this point is within a distance of  $r$  of our line  $L$ , we have that  $D < \sqrt{R^2 + r^2} < \sqrt{R^2 + 1}$ . So we have that  $R < D < \sqrt{R^2 + 1}$ . Then  $R^2 < D^2 < R^2 + 1$ , but  $D^2$  is an integer so this is impossible.

**Problem 23: Challenge**

Prove that there exists a rational approximation of  $\sqrt{3}$  within  $10^{-3}$  with denominator at most 501. Come up with an upper bound for the smallest denominator of a  $\epsilon$ -close rational approximation of any irrational number  $\alpha > 0$ . Your bound can have some dependence on  $\alpha$  and should get smaller as  $\alpha$  gets larger.

*Hint:* Use the orchard.

**Solution**

Take the line through the origin of slope  $\sqrt{3}$ . We would like an orchard for which  $r = 10^{-3}$  gives no line of sight, since this will guarantee an integer point within a distance of  $10^{-3}$ . Then by our previous problem, we can take  $10^{-3} > \frac{1}{R}$ , so take  $R > 1000$ . Now since this line intersects the boundary of the orchard at  $(\frac{R}{2}, \frac{\sqrt{3}R}{2})$ , we have that the  $x$ -coordinate is at most  $\frac{R}{2} = 501$ . Then we have that our lattice point  $(x, y)$  satisfies  $\sqrt{3}x - 10^{-3} < y < \sqrt{3}x + 10^{-3}$ , so  $\sqrt{3} - 10^{-3} < \frac{y}{x} < \sqrt{3} + \frac{10^{-3}}{x}$ . Therefore,  $\frac{y}{x}$  is a rational approximation that is  $10^{-3}$ -close to  $\sqrt{3}$  and has denominator at most 501. Notice that we got closer than we need to. Repeating this same process, our upper bound for the denominator of an  $\epsilon$ -close approximation of  $\alpha$  is  $\frac{\cos(\text{atan}(\alpha))}{\epsilon} = \frac{1}{\sqrt{\alpha^2 + 1}\epsilon}$ .