Pidgeonhole Problems

Prepared by Mark on February 13, 2025

Problem 1:
Is it possible to cover an equilateral triangle with two smaller equilateral triangles? Why or why not?

Problem 2:
You are given $n+1$ distinct integers.
Prove that at least two of them have a difference divisible by n.

Problem 3:
You have an 8×8 chess board with two opposing corner squares cut off. You also have a set of dominoes, each of which is the size of two squares. Is it possible to completely cover the board with dominos, so that none overlap nor stick out?

Problem 4:
The ocean covers more than a half of the Earth's surface. Prove that the ocean has at least one pair of antipodal points.

Problem 5:
There are $n > 1$ people at a party. Prove that among them there are at least two people who have the same number of acquaintances at the gathering. (We assume that if A knows B, then B also knows A)

Problem 6:
Pick five points in $\mathbb{R}^2$ with integral coordinates. Show that two of these form a line segment that has an integral midpoint.

Problem 7:
Every point on a line is painted black or white. Show that there exist three points of the same color where one is the midpoint of the line segment formed by the other two.

Problem 8:
Every point on a plane is painted black or white. Show that there exist two points in the plane that have the same color and are located exactly one foot away from each other.

Problem 9:
Let n be an integer not divisible by 2 and 5. Show that n has a multiple consisting entirely of ones.

Problem 10:
Prove that for any $n > 1$, there exists an integer made of only sevens and zeros that is divisible by n.

Problem 11:
Choose $n + 1$ integers between 1 and $2n$. Show that at least two of these are co-prime.

Problem 12:
Choose $n + 1$ integers between 1 and $2n$. Show that you must select two numbers a and b such that a divides b.

Problem 13:
Show that it is always possible to choose a subset of the set of integers $\{a_1, a_2, ..., a_n\}$ so that the sum of the numbers in the subset is divisible by n.

Problem 14:
Show that there exists a positive integer divisible by 2013 that has 2014 as its last four digits.

Problem 15:
Let n be an odd number. Let $a_1, a_2, ..., a_n$ be a permutation of the numbers $1, 2, ..., n$.
Show that $(a_1 - 1) \times (a_2 - 2) \times ... \times (a_n - n)$ is even.

Problem 16:
A stressed-out student consumes at least one espresso every day of a particular year, drinking 500 overall. Show the student drinks exactly 100 espressos on some consecutive sequence of days.

Problem 17:
Show that there are either three mutual acquaintances or four mutual strangers at a party with ten or more people.

Problem 18:
Given a table with a marked point, O, and with 2013 properly working watches put down on the table, prove that there exists a moment in time when the sum of the distances from O to the watches' centers is less than the sum of the distances from O to the tips of the watches' minute hands.