

# Continued Fractions

Prepared by Mark on February 13, 2025

Based on a handout by Matthew Gherman and Adam Lott

## Instructor's Handout

### Part 1: The Euclidean Algorithm

#### Definition 1:

The *greatest common divisor* of  $a$  and  $b$  is the greatest integer that divides both  $a$  and  $b$ . We denote this number with  $\gcd(a, b)$ . For example,  $\gcd(45, 60) = 15$ .

#### Problem 2:

Find  $\gcd(20, 14)$  by hand.

#### Solution

$$\gcd(20, 14) = 2$$

#### Theorem 3: The Division Algorithm

Given two integers  $a, b$ , we can find two integers  $q, r$ , where  $0 \leq r < b$  and  $a = qb + r$ . In other words, we can divide  $a$  by  $b$  to get  $q$  remainder  $r$ .

For example, take  $14 \div 3$ . We can re-write this as  $3 \times 4 + 2$ .

Here,  $a$  and  $b$  are 14 and 3,  $q = 4$  and  $r = 2$ .

#### Theorem 4:

For any integers  $a, b, c$ ,

$$\gcd(ac + b, a) = \gcd(a, b)$$

#### Problem 5:

Compute  $\gcd(668, 6)$ . *Hint:*  $668 = 111 \times 6 + 2$

Then, compute  $\gcd(3 \times 668 + 6, 668)$ .

#### Problem 6: The Euclidean Algorithm

Using the two theorems above, detail an algorithm for finding  $\gcd(a, b)$ .

Then, compute  $\gcd(1610, 207)$  by hand.

#### Solution

Using Theorem 4 and the division algorithm,

$$\begin{aligned} \gcd(1610, 207) &= \gcd(207, 161) & 1610 &= 207 \times 7 + 161 \\ &= \gcd(161, 46) & 207 &= 161 \times 1 + 46 \\ &= \gcd(46, 23) & 161 &= 46 \times 3 + 23 \\ &= \gcd(23, 0) = 23 & 46 &= 23 \times 2 + 0 \end{aligned}$$

## Part 2:

**Definition 7:**

A *finite continued fraction* is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_{k-1} + \frac{1}{a_k}}}}}$$

where  $a_0, a_1, \dots, a_k$  are all in  $\mathbb{Z}_0^+$ . We'll denote this as  $[a_0, a_1, \dots, a_k]$ .

**Problem 8:**

Write each of the following as a continued fraction.

*Hint:* Solve for one  $a_n$  at a time.

- $5/12$
- $5/3$
- $33/23$
- $37/31$

**Problem 9:**

Write each of the following continued fractions as a regular fraction in lowest terms:

- $[2, 3, 2]$
- $[1, 4, 6, 4]$
- $[2, 3, 2, 3]$
- $[9, 12, 21, 2]$

**Problem 10:**

Let  $\frac{p}{q}$  be a positive rational number in lowest terms. Perform the Euclidean algorithm to obtain the following sequence:

$$\begin{aligned} p &= q_0q + r_1 \\ q &= q_1r_1 + r_2 \\ r_1 &= q_2r_2 + r_3 \\ &\vdots \\ r_{k-1} &= q_kr_k + 1 \\ r_k &= q_{k+1} \end{aligned}$$

We know that we will eventually get 1 as the remainder because  $p$  and  $q$  are relatively prime. Show that  $p/q = [q_0, q_1, \dots, q_{k+1}]$ .

**Problem 11:**

Repeat Problem 8 using the method outlined in Problem 10.

**Definition 12:**

An *infinite continued fraction* is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

where  $a_0, a_1, a_2, \dots$  are in  $\mathbb{Z}_0^+$ . To prove that this expression actually makes sense and equals a finite number is beyond the scope of this worksheet, so we assume it for now. This is denoted  $[a_0, a_1, a_2, \dots]$ .

**Problem 13:**

Using a calculator, compute the first five terms of the continued fraction expansion of the following numbers. Do you see any patterns?

- $\sqrt{2}$
- $\pi \approx 3.14159\dots$
- $\sqrt{5}$
- $e \approx 2.71828\dots$

**Problem 14:**

Show that an  $\alpha \in \mathbb{R}^+$  can be written as a finite continued fraction if and only if  $\alpha$  is rational.

*Hint:* For one of the directions, use Problem 10

**Definition 15:**

The continued fraction  $[a_0, a_1, a_2, \dots]$  is *periodic* if it ends in a repeating sequence of digits. A few examples are below. We denote the repeating sequence with a line.

- $[1, 2, 2, 2, \dots] = [1, \overline{2}]$  is periodic.
- $[1, 2, 3, 4, 5, \dots]$  is not periodic.
- $[1, 3, 7, 6, 4, 3, 4, 3, 4, 3, \dots] = [1, 3, 7, 6, \overline{4, 3}]$  is periodic.
- $[1, 2, 4, 8, 16, \dots]$  is not periodic.

**Problem 16:**

- Show that  $\sqrt{2} = [1, \overline{2}]$ .
- Show that  $\sqrt{5} = [1, \overline{4}]$ .

*Hint:* use the same strategy as Problem 13 but without a calculator.

**Problem 17: Challenge I**

Express the following continued fractions in the form  $\frac{a+\sqrt{b}}{c}$  where  $a$ ,  $b$ , and  $c$  are integers:

- $[\overline{1}]$
- $[\overline{2, 5}]$
- $[1, 3, \overline{2, 3}]$

**Problem 18: Challenge II**

Let  $\alpha = [a_0, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+p}}]$  be any periodic continued fraction.

Prove that  $\alpha$  is of the form  $\frac{a+\sqrt{b}}{c}$  for some integers  $a, b, c$  where  $b$  is not a perfect square.

**Problem 19: Challenge III**

Prove that any number of the form  $\frac{a+\sqrt{b}}{c}$  where  $a, b, c$  are integers and  $b$  is not a perfect square can be written as a periodic continued fraction.

## Part 3: Convergents

**Definition 20:**

Let  $\alpha = [a_0, a_1, a_2, \dots]$  be an infinite continued fraction (aka an irrational number).

The  $n$ th convergent to  $\alpha$  is the rational number  $[a_0, a_1, \dots, a_n]$  and is denoted  $C_n(\alpha)$ .

**Problem 21:**

Calculate the following convergents and write them in lowest terms:

- $C_3([1, 2, 3, 4, \dots])$
- $C_4([0, \overline{2, 3}])$
- $C_5([1, \overline{5}])$

**Problem 22:**

Recall from last week that  $\sqrt{5} = [2, \overline{4}]$ . Calculate the first five convergents to  $\sqrt{5}$  and write them in lowest terms. Do you notice any patterns?

*Hint:* Look at the numbers  $\sqrt{5} - C_j(\sqrt{5})$  for  $1 \leq j \leq 5$

**Properties of Convergents**

In this section, we want to show that the  $n$ th convergent to a real number  $\alpha$  is the best approximation of  $\alpha$  with the given denominator. Let  $\alpha = [a_0, a_1, \dots]$  be fixed, and we will write  $C_n$  instead of  $C_n(\alpha)$  for short. Let  $p_n/q_n$  be the expression of  $C_n$  as a rational number in lowest terms. We will eventually prove that  $|\alpha - C_n| < \frac{1}{q_n^2}$ , and there is no better rational estimate of  $\alpha$  with denominator less than or equal to  $q_n$ .

First we want the recursive formulas  $p_n = a_n p_{n-1} + p_{n-2}$  and  $q_n = a_n q_{n-1} + q_{n-2}$  given  $p_{-1} = 1$ ,  $p_0 = a_0$ ,  $q_{-1} = 0$ , and  $q_0 = 1$ .

**Problem 23:**

Verify the recursive formula for  $1 \leq j \leq 3$  for the convergents  $C_j$  of:

- $[1, 2, 3, 4, \dots]$
- $[0, \overline{2, 3}]$
- $[\overline{1, 5}]$

**Problem 24: Challenge IV**

Prove that  $p_n = a_n p_{n-1} + p_{n-2}$  and  $q_n = a_n q_{n-1} + q_{n-2}$  by induction.

- As the base case, verify the recursive formulas for  $n = 1$  and  $n = 2$ .
- Assume the recursive formulas hold for  $n \leq m$  and show the formulas hold for  $m + 1$ .

**Problem 25:**

Using the recursive formula from Problem 24, we will show that  $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ .

- What is  $p_1 q_0 - p_0 q_1$ ?
- Substitute  $a_n p_{n-1} + p_{n-2}$  for  $p_n$  and  $a_n q_{n-1} + q_{n-2}$  for  $q_n$  in  $p_n q_{n-1} - p_{n-1} q_n$ . Simplify the expression.
- What happens when  $n = 2$ ? Explain why  $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ .

**Problem 26: Challenge VI**

Similarly derive the formula  $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$ .

**Problem 27:**

Recall  $C_n = p_n/q_n$ . Show that  $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1}q_n}$  and  $C_n - C_{n-2} = \frac{(-1)^{n-2}a_n}{q_{n-2}q_n}$ .

*Hint:* Use Problem 25 and  $p_nq_{n-2} - p_{n-2}q_n = (-1)^{n-2}a_n$  respectively

In Problem 22, the value  $\alpha - C_n$  alternated between negative and positive and  $|\alpha - C_n|$  got smaller each step. Using the relations in Problem 27, we can prove that this is always the case. Specifically,  $\alpha$  is always between  $C_n$  and  $C_{n+1}$ .

**Problem 28:**

Let's figure out how well the  $n$ th convergents estimate  $\alpha$ . We will show that  $|\alpha - C_n| < \frac{1}{q_n^2}$ .

- Note that  $|C_{n+1} - C_n| = \frac{1}{q_nq_{n+1}}$ .
- Why is  $|\alpha - C_n| \leq |C_{n+1} - C_n|$ ?
- Conclude that  $|\alpha - C_n| < \frac{1}{q_n^2}$ .

We are now ready to prove a fundamental result in the theory of rational approximation.

**Problem 29: Dirichlet's approximation theorem**

Let  $\alpha$  be any irrational number. Prove that there are infinitely many rational numbers  $\frac{p}{q}$  such that  $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ .



**Problem 30: Challenge VII**

Prove that if  $\alpha$  is *rational*, then there are only *finitely* many rational numbers  $\frac{p}{q}$  satisfying

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

The above result shows that the  $n$ th convergents estimate  $\alpha$  extremely well. Are there better estimates for  $\alpha$  if we want small denominators? In order to answer this question, we introduce the Farey sequence.

**Definition 31:**

The *Farey sequence* of order  $n$  is the set of rational numbers between 0 and 1 whose denominators (in lowest terms) are  $\leq n$ , arranged in increasing order.

**Problem 32:**

List the Farey sequence of order 4. Now figure out the Farey sequence of order 5 by including the relevant rational numbers in the Farey sequence of order 4.

**Problem 33:**

Let  $\frac{a}{b}$  and  $\frac{c}{d}$  be consecutive elements of the Farey sequence of order 5. What does  $bc - ad$  equal?

**Problem 34: Challenge VIII**

Prove that  $bc - ad = 1$  for  $\frac{a}{b}$  and  $\frac{c}{d}$  consecutive rational numbers in Farey sequence of order  $n$ .

- In the plane, draw the triangle with vertices  $(0,0)$ ,  $(b,a)$ ,  $(d,c)$ . Show that the area  $A$  of this triangle is  $\frac{1}{2}$  using Pick's Theorem. Recall that Pick's Theorem states  $A = \frac{B}{2} + I - 1$  where  $B$  is the number of lattice points on the boundary and  $I$  is the number of points in the interior.

*Hint:*  $B=3$  and  $I=0$

- Show that the area of the triangle is also given by  $\frac{1}{2}|ad - bc|$ .
- Why is  $bc > ad$ ?
- Conclude that  $bc - ad = 1$ .

**Problem 35:**

Use the result of Problem 34 to show that there is no rational number between  $C_{n-1}$  and  $C_n$  with denominator less than or equal to  $q_n$ . Conclude that if  $a/b$  is any rational number with  $b \leq q_n$ , then  $|\alpha - \frac{a}{b}| \geq |\alpha - \frac{p_n}{q_n}|$

**Problem 36: Challenge X**

Prove the following strengthening of Dirichlet's approximation theorem. If  $\alpha$  is irrational, then there are infinitely many rational numbers  $\frac{p}{q}$  satisfying  $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$ .

- Prove that  $(x + y)^2 \geq 4xy$  for any real  $x, y$ .
- Let  $p_n/q_n$  be the  $n$ th convergent to  $\alpha$ . Prove that

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right|^2 \geq 4 \left| \frac{p_n}{q_n} - \alpha \right| \left| \frac{p_{n+1}}{q_{n+1}} - \alpha \right|$$

*Hint:*  $\alpha$  lies in between  $\frac{p_n}{q_n}$  and  $\frac{p_{n+1}}{q_{n+1}}$

- Prove that either  $\frac{p_n}{q_n}$  or  $\frac{p_{n+1}}{q_{n+1}}$  satisfies the desired inequality (Hint: proof by contradiction).
- Conclude that there are infinitely many rational numbers  $\frac{p}{q}$  satisfying  $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$ .