Fast Inverse Square Root

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Instructor's Handout
This handout contains solutions and notes.
Recompile without solutions before distributing.

Section 1: Introduction

In 2005, ID Software published the source code of *Quake III Arena*, a popular game released in 1999. This caused quite a stir: ID Software was responsible for many games popular among old-school engineers (most notably *Doom*, which has a place in programmer humor even today).

Naturally, this community immediately began dissecting *Quake*'s source.
One particularly interesting function is reproduced below, with original comments:

```
float Q_rsqrt( float number ) {
  long i;
  float x2, y;
  const float threehalfs = 1.5F;

  x2 = number * 0.5F;
  y  = number;
  i  = * ( long * ) &y;           // evil floating point bit level hacking
  i  = 0x5f3759df - ( i >> 1 );               // [redacted]
  y  = * ( float * ) &i;
  y  = y * ( threehalfs - ( x2 * y * y ) );   // 1st iteration
//  y  = y * ( threehalfs - ( x2 * y * y ) );   // 2nd iteration, this can be removed

  return y;
}
```

This code defines a function `Q_sqrt`, which was used as a fast approximation of the inverse square root in graphics routines. (in other words, `Q_sqrt` efficiently approximates $1 \div \sqrt{x}$)

The key word here is "fast": *Quake* ran on very limited hardware, and traditional approximation techniques (like Taylor series)[1] were too computationally expensive to be viable.

Our goal today is to understand how `Q_sqrt` works.
To do that, we'll first need to understand how computers represent numbers.
We'll start with simple binary integers—turn the page.

[1] Taylor series aren't used today, and for the same reason. There are better ways.

Section 2: Integers

Definition 1:
A *bit string* is a string of binary digits.
In this handout, we'll denote bit strings with the prefix `0b`.
This prefix is only notation—it is *not* part of the string itself.
For example, 1001 is the number "one thousand and one," while `0b1001` is the string of bits "1 0 0 1".

We will separate long bit strings with underscores for readability.
Underscores have no meaning: `0b1111_0000` = `0b11110000`.

Problem 2:
What is the value of the following bit strings, if we interpret them as integers in base 2?
- `0b0001_1010`
- `0b0110_0001`

Solution:
- `0b0001_1010` $= 2 + 8 + 16 = 26$
- `0b0110_0001` $= 1 + 32 + 64 = 95$

Definition 3:
We can interpret a bit string in any number of ways.
One such interpretation is the *unsigned integer*, or `uint` for short.
`uint`s allow us to represent positive (hence "unsigned") integers using 32-bit strings.

The value of a `uint` is simply its value as a binary number:
- `0b00000000_00000000_00000000_00000000` $= 0$
- `0b00000000_00000000_00000000_00000011` $= 3$
- `0b00000000_00000000_00000000_00100000` $= 32$
- `0b00000000_00000000_00000000_10000010` $= 130$

Problem 4:
What is the largest number we can represent with a 32-bit `uint`?

Solution:
`0b11111111_11111111_11111111_11111111` $= 2^{32} - 1$

Problem 5:
Find the value of each of the following 32-bit unsigned integers:

- `0b00000000_00000000_00000101_00111001`
- `0b00000000_00000000_00000001_00101100`
- `0b00000000_00000000_00000100_10110000`

Hint: The third conversion is easy—look carefully at the second.

Instructor note:

Consider making a list of the powers of two ≥ 1024 on the board.

Solution:

- `0b00000000_00000000_00000101_00111001` $= 1337$
- `0b00000000_00000000_00000001_00101100` $= 300$
- `0b00000000_00000000_00000010_01011000` $= 1200$

Notice that the third int is the second shifted left twice (i.e, multiplied by 4)

Definition 6:
In general, fast division of `uints` is difficult[2].
Division by powers of two, however, is incredibly easy:
To divide by two, all we need to do is shift the bits of our integer right.

For example, consider `0b0000_0110` $= 6$.
If we insert a zero at the left end of this string and delete the zero at the right (thus "shifting" each bit right), we get `0b0000_0011`, which is 3.

Of course, we lose the remainder when we right-shift an odd number:
9 shifted right is 4, since `0b0000_1001` shifted right is `0b0000_0100`.

Problem 7:
Right shifts are denoted by the `>>` symbol:
`00110 >>` n means "shift `0b0110` right n times."
Find the value of the following:

- 12 >> 1
- 27 >> 3
- 16 >> 8

Naturally, you'll have to convert these integers to binary first.

Solution:

- 12 >> 1 = 6
- 27 >> 3 = 3
- 16 >> 8 = 0

[2] One may use repeated subtraction, but this isn't efficient.

Section 3: Floats

Definition 8:
Binary decimals[3] are very similar to base-10 decimals.
In base 10, we interpret place value as follows:

- $0.1 = 10^{-1}$
- $0.03 = 3 \times 10^{-2}$
- $0.0208 = 2 \times 10^{-2} + 8 \times 10^{-4}$

We can do the same in base 2:

- $\texttt{0.1} = 2^{-1} = 0.5$
- $\texttt{0.011} = 2^{-2} + 2^{-3} = 0.375$
- $\texttt{101.01} = 5.125$

Problem 9:
Rewrite the following binary decimals in base 10:
You may leave your answer as a fraction.

- `1011.101`
- `110.1101`

[3]Note that "binary decimal" is a misnomer—"deci" means "ten"!

Definition 10:
Another way we can interpret a bit string is as a *signed floating-point decimal*, or a `float` for short.
Floats represent a subset of the real numbers, and are interpreted as follows:
The following only applies to floats that consist of 32 bits. We won't encounter any others today.

$$\texttt{0b}\underbrace{\texttt{0}}_{\text{s}}\texttt{\_}\underbrace{\texttt{00000000}}_{\text{exponent}}\texttt{\_}\underbrace{\texttt{0000000\_00000000\_00000000}}_{\text{fraction}}$$

- The first bit denotes the sign of the float's value We'll label it s.
  If $s = \texttt{1}$, this float is negative; if $s = \texttt{0}$, it is positive.
- The next eight bits represent the *exponent* of this float. (we'll see what that means soon)
  We'll call the value of this eight-bit binary integer E.
  Naturally, $0 \leq E \leq 255$ (since E consist of eight bits)
- The remaining 23 bits represent the *fraction* of this float.
  They are interpreted as the fractional part of a binary decimal.
  For example, the bits `0b10100000_00000000_00000000` represent $0.5 + 0.125 = 0.625$.
  We'll call the value of these bits as a binary integer F.
  Their value as a binary decimal is then $F \div 2^{23}$. (convince yourself of this)

Problem 11:
Consider `0b01000001_10101000_00000000_00000000`.
Hint: The underscores here do *not* match those in Definition 10

Find the s, E, and F we get if we interpret this bit string as a `float`.
Leave F as a sum of powers of two.

Solution:

$s = 0$
$E = 131$
$F = 2^{21} + 2^{19}$

Definition 12:
The final value of a float with sign s, exponent E, and fraction F is

$$(-1)^s \times 2^{E-127} \times \left(1 + \frac{F}{2^{23}}\right)$$

Notice that this is very similar to base-10 scientific notation, which is written as

$$(-1)^s \times 10^e \times (f)$$

We subtract 127 from E so we can represent positive and negative numbers.
E is an eight bit binary integer, so $0 \leq E \leq 255$ and thus $-127 \leq (E - 127) \leq 127$.

Problem 13:
Consider `0b01000001_10101000_00000000_00000000`.
This is the same bit string we used in Problem 11.

What value do we get if we interpret this bit string as a float?
Hint: $21 \div 16 = 1.3125$

Solution:

This is 21:

$$2^4 \times \left(1 + \frac{2^{21} + 2^{19}}{2^{23}}\right) = 2^4 \times \left(1 + 2^{-2} + 2^{-4}\right) = 16 + 4 + 1 = 21$$

Problem 14:
Encode 12.5 as a float.
Hint: $12.5 \div 8 = 1.5625$

Solution:

$$12.5 = 8 \times 1.5625 = 2^3 \times (1 + (0.5 + 0.0625)) = 2^{130} \times \left(1 + \frac{2^{22} + 2^{19}}{2^{23}}\right)$$

which is `0b01000001_01001000_00000000_00000000`.

Definition 15:
Say we have a bit string x.
We'll let x_f denote the value we get if we interpret x as a float,
and we'll let x_i denote the value we get if we interpret x an integer.

Problem 16:
Let $x =$ `0b01000001_01001000_00000000_00000000`.
What are x_f and x_i? As always, you may leave big numbers as powers of two.

Solution:

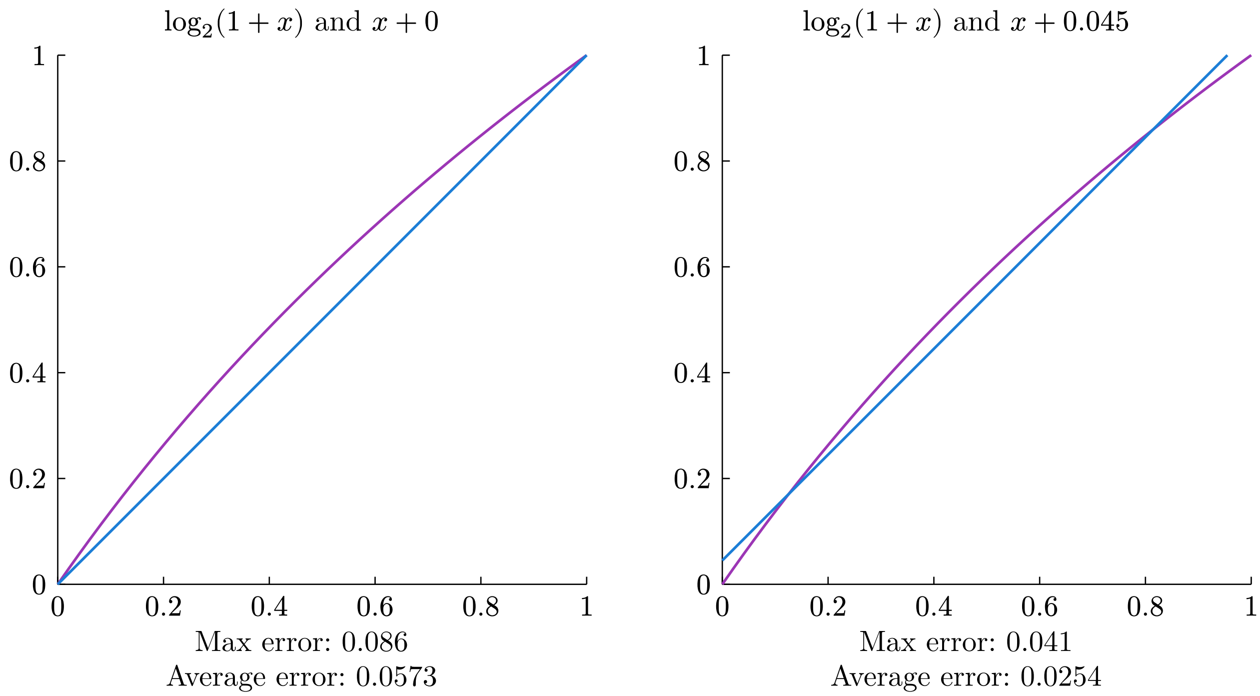
$x_f = 12.5$

$x_i = 2^{30} + 2^{24} + 2^{22} + 2^{19} = 11,095,237,632$

Section 4: Integers and Floats

Observation:
If x is smaller than 1, $\log_2(1+x)$ is approximately equal to x.
Note that this equality is exact for $x = 0$ and $x = 1$, since $\log_2(1) = 0$ and $\log_2(2) = 1$.

We'll add the *correction term* ε to our approximation: $\log_2(1+a) \approx a + \varepsilon$.
This allows us to improve the average error of our linear approximation:



A suitiable value of ε can be found using calculus or with computational trial-and-error.
We won't bother with this—we'll simply leave the correction term as an opaque constant ε.

Note: "Average error" above is simply the area of the region between the two graphs:

$$\int_0^1 |\ \log(1+x)_2 - (x+\varepsilon)\ |$$

Feel free to ignore this note, it isn't a critical part of this handout.

Problem 17:
Use the fact that $\log_2(1+a) \approx a + \varepsilon$ to approximate $\log_2(x_f)$ in terms of x_i.
Namely, show that

$$\log_2(x_f) = \frac{x_i}{2^{23}} - 127 + \varepsilon$$

In other words, we're finding an expression for x as a float in terms of x as an int.

Solution:

Let E and F be the exponent and float bits of x_f.
We then have:

$$\begin{aligned}
\log_2(x_f) &= \log_2\left(2^{E-127} \times \left(1 + \frac{F}{2^{23}}\right)\right) \\
&= E - 127 + \log_2\left(1 + \frac{F}{2^{23}}\right) \\
&\approx E - 127 + \frac{F}{2^{23}} + \varepsilon \\
&= \frac{1}{2^{23}}(2^{23}E + F) - 127 + \varepsilon \\
&= \frac{1}{2^{23}}(x_i) - 127 + \varepsilon
\end{aligned}$$

Problem 18:
Using basic log rules, rewrite $\log_2\left(\frac{1}{\sqrt{x}}\right)$ in terms of $\log_2(x)$.

Solution:

$$\log_2\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2}\log_2(x)$$

Section 5: The Fast Inverse Square Root

A simplified version of the *Quake* routine we are studying is reproduced below.

```
float Q_rsqrt( float number ) {
    long i = * ( long * ) &number;
    i = 0x5f3759df - ( i >> 1 );
    return * ( float * ) &i;
}
```

This code defines a function `Q_rsqrt` that consumes a float `number` and approximates its inverse square root. If we rewrite this using notation we're familiar with, we get the following:

$$\texttt{Q\_sqrt}\left(n_f\right) = 6240089 - (n_i \div 2) \qquad \approx \frac{1}{\sqrt{n_f}}$$

`0x5f3759df` is 6240089 in hexadecimal.
Ask an instructor to explain if you don't know what this means.
It is a magic number hard-coded into `Q_sqrt`.

Our goal in this section is to understand why this works:

- How does Quake approximate $\frac{1}{\sqrt{x}}$ by simply subtracting and dividing by two?
- What's special about 6240089?

Remark 19:
For those that are interested, here are the details of the "code-to-math" translation:

- "`long i = * (long *) &number`" is C magic that tells the compiler
  to set `i` to the `uint` value of the bits of `number`.
  "long" refers to a "long integer", which has 32 bits.
  Normal `int`s have 16 bits, `short int`s have 8.
  In other words, `number` is n_f and `i` is n_i.

- Notice the right-shift in the second line of the function.
  We translated `(i >> 1)` into $(n_i \div 2)$.

- "`return * (float *) &i`" is again C magic.
  Much like before, it tells us to return the value of the bits of `i` as a float.

Setup:
We are now ready to show that `Q_sqrt` (x) effectively approximates $\frac{1}{\sqrt{x}}$.
For convenience, let's call the bit string of the inverse square root r.
In other words,

$$r_f := \frac{1}{\sqrt{n_f}}$$

This is the value we want to approximate.

Problem 20:
Find an approximation for $\log_2(r_f)$ in terms of n_i and ε
Remember, ε is the correction constant in our approximation of $\log_2(1+x)$.

Solution:

$$\log_2(r_f) = \log_2\left(\frac{1}{\sqrt{n_f}}\right) = \frac{-1}{2}\log_2(n_f) \approx \frac{-1}{2}\left(\frac{n_i}{2^{23}} + \varepsilon - 127\right)$$

Problem 21:
Let's call the "magic number" in the code above κ, so that

$$\texttt{Q\_sqrt}\,(n_f) = \kappa - (n_i \div 2)$$

Use Problem 17 and Problem 20 to show that `Q_sqrt` $(n_f) \approx r_i$
Note: If we know r_i, we know r_f.
We don't even need to convert between the two—the underlying bits are the same!

Solution:

From Problem 17, we know that

$$\log_2(r_f) \approx \frac{r_i}{2^{23}} + \varepsilon - 127$$

Combining this with the result from Problem 20, we get:

$$\begin{aligned}
\frac{r_i}{2^{23}} + \varepsilon - 127 &\approx \frac{-1}{2}\left(\frac{n_i}{2^{23}} + \varepsilon - 127\right) \\
\frac{r_i}{2^{23}} &\approx \frac{-1}{2}\left(\frac{n_i}{2^{23}}\right) + \frac{3}{2}(127 - \varepsilon) \\
r_i &\approx \frac{-1}{2}(n_i) + 2^{23}\frac{3}{2}(127 - \varepsilon) = 2^{23}\frac{3}{2}(127 - \varepsilon) - \frac{n_i}{2}
\end{aligned}$$

This is exactly what we need! If we set κ to $(3 \times 2^{22})(127 - \varepsilon)$, then

$$r_i \approx \kappa - (n_i \div 2) = \texttt{Q\_sqrt}\,(n_f)$$

Problem 22:
What is the exact value of κ in terms of ε?
Hint: Look at Problem 21. We already found it!

Solution:

This problem makes sure our students see that $\kappa = (3 \times 2^{22})(127 - \varepsilon)$.
See the solution to Problem 21.

Remark 23:
In Problem 22 we saw that $\kappa = (3 \times 2^{22})(127 - \varepsilon)$.
Looking at the code again, we see that $\kappa =$ `0x5f3759df` in *Quake*:

```
float Q_rsqrt( float number ) {
    long i = * ( long * ) &number;
    i = 0x5f3759df - ( i >> 1 );
    return * ( float * ) &i;
}
```

Using a calculator and some basic algebra, we can find the ε this code uses:
Remember, `0x5f3759df` is 6240089 in hexadecimal.

$$\begin{aligned}(3 \times 2^{22})(127 - \varepsilon) &= 6240089 \\ (127 - \varepsilon) &= 126.955 \\ \varepsilon &= 0.0450466\end{aligned}$$

So, 0.045 is the ε used by Quake.
Online sources state that this constant was generated by trial-and-error,
though it is fairly close to the ideal ε.

Remark 24:
And now, we're done!
We've shown that `Q_sqrt(x)` approximates $\frac{1}{\sqrt{x}}$ fairly well.

Notably, `Q_sqrt` uses *zero* divisions or multiplications (`>>` doesn't count).
This makes it *very* fast when compared to more traditional approximation techniques (i.e, Taylor series).

In the case of *Quake*, this is very important. 3D graphics require thousands of inverse-square-root calculations to render a single frame[4], which is not an easy task for a Playstation running at 300MHz.

Instructor note:

Let x be a bit string. If we assume x_f is positive and E is even, then

$$(x \texttt{ >> } 1)_f = 2^{(E \div 2) - 127} \times \left(1 + \frac{F \div 2}{2^{23}}\right)$$

Notably: a right-shift divides the exponent of x_f by two,
which is, of course, a square root!

This intuition is hand-wavy, though:
If E is odd, its lowest-order bit becomes the highest-order bit of F when we shift x right.
Also, a right shift doesn't divide the *entire* exponent, skipping the -127 offset.

Remarkably, this intuition is still somewhat correct.
The bits align *just so*, and our approximation still works.

One can think of the fast inverse root as a "digital slide rule":
The integer representation of x_f already contains $\log_2(x_f)$, offset and scaled.
By subtracting and dividing in "log space", we effectively invert and root x_f!
After all,

$$-\frac{1}{2}\log_2(n_f) = \frac{1}{\sqrt{n_f}}$$

[4]e.g, to generate normal vectors

Section 6: Bonus – More about Floats

Problem 25:
Convince yourself that all numbers that can be represented as a float are rational.

Problem 26:
Find a rational number that cannot be represented as a float.

Problem 27:
What is the smallest positive 32-bit float?

Problem 28:
What is the largest positive 32-bit float?

Problem 29:
How many floats are between -1 and 1?

Problem 30:
How many floats are between 1 and 2?

Problem 31:
How many floats are between 1 and 128?