
Lattices

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Instructor's Handout

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Definition 1:

The *integer lattice* \mathbb{Z}^n is the set of points with integer coordinates in n dimensions. For example, \mathbb{Z}^3 is the set of points (a, b, c) where a , b , and c are integers.

Problem 2:

Draw \mathbb{Z}^2 .

Definition 3:

We say a set of vectors $\{v_1, v_2, \dots, v_k\}$ *generates* \mathbb{Z}^n if every lattice point can be written as

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k$$

for integer coefficients a_i .

Bonus: show that k must be at least n .

Problem 4:

Which of the following generate \mathbb{Z}^2 ?

- $\{(1, 2), (2, 1)\}$
- $\{(1, 0), (0, 2)\}$
- $\{(1, 1), (1, 0), (0, 1)\}$

Solution

Only the last.

Problem 5:

Find a set of two vectors other than $\{(0, 1), (1, 0)\}$ that generates \mathbb{Z}^2 .

Problem 6:

Find a set of vectors that generates \mathbb{Z}^n .

Definition 7:

Say we have a generating set of a lattice.

The *fundamental region* of this set is the n -dimensional parallelogram spanned by its members.

Problem 8:

Draw two fundamental regions of \mathbb{Z}^2 using two different generating sets. Verify that their volumes are the same.

Part 1: Minkowski's Theorem

Theorem 9: Blichfeldt's Theorem

Let X be a finite connected region. If the volume of X is greater than 1, X must contain two distinct points that differ by an element of \mathbb{Z}^n . In other words, there exist distinct $x, y \in X$ so that $x - y \in \mathbb{Z}^n$.

Intuitively, this means that you can translate X to cover two lattice points at the same time.

Problem 10:

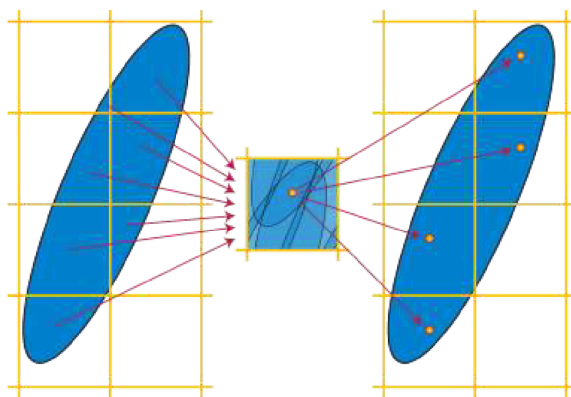
Draw a connected region in \mathbb{R}^2 with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint:* Area is two-dimensional volume.

Problem 11:

Draw a *disconnected* region in \mathbb{R}^2 with volume greater than 1 that contains no lattice points, and show that no two points in that region differ by an integer vector. In other words, show that Theorem 9 indeed requires a connected region.

Problem 12:

The following picture gives an idea for the proof of Blichfeldt's theorem in \mathbb{Z}^2 . Explain the picture and complete the proof.



Solution

The fundamental region of \mathbb{Z}^2 tiles the plane. Translate these tiles by lattice vectors to stack them on the fundamental region. Then since the union of the intersections of X with these tiles has area greater 1 and they are stacked on a region of area 1, there must be an overlap by a generalization of the pigeonhole principle (if there were no overlap then the sum of the areas would be less than or equal to 1). Take points x, y in the overlap. Then $x - y$ is a lattice point corresponding to the difference in translates, which were lattice points. Hence, $x - y \in \mathbb{Z}^2$.

Problem 13:

Let X be a region $\in \mathbb{R}^2$ of volume k . How many integral points must X contain after a translation?

Solution

$$\lceil k \rceil$$

Definition 14:

A region X is *convex* if the line segment connecting any two points in X lies entirely in X .

Problem 15:

Draw a convex region in two dimensions.

Then, draw a two-dimensional region that is *not* convex.

Definition 16:

We say a region X is *symmetric with respect to the origin* if for all points $x \in X$, $-x$ is also in X . In the following problems, “*symmetric*” means “symmetric with respect to the origin.”

Problem 17:

Draw a symmetric region.

Then, draw an asymmetric region.

Problem 18:

Show that a convex symmetric set always contains the origin.

Theorem 19: Minkowski’s Theorem

Every convex set in \mathbb{R}^n that is symmetric and has a volume greater than 2^n contains an integral point that isn’t zero.

Problem 20:

Draw a few sets that satisfy Theorem 19 in \mathbb{R}^2 .

What is a simple class of regions that has the properties listed above?

Problem 21:

Let K be a region in \mathbb{R}^2 satisfying Theorem 19.

Let K' be this region scaled by $\frac{1}{2}$.

- How does the volume of K' compare to K ?
- Show that the sum of any two points in K' lies in K *Hint: Use convexity.*
- Apply Blichfeldt’s theorem to K' to prove Minkowski’s theorem in \mathbb{R}^2 .

Problem 22:

Let K be a region in \mathbb{R}^n satisfying Theorem 19. Scale this region by $\frac{1}{2}$, called $K' = \frac{1}{2}K$.

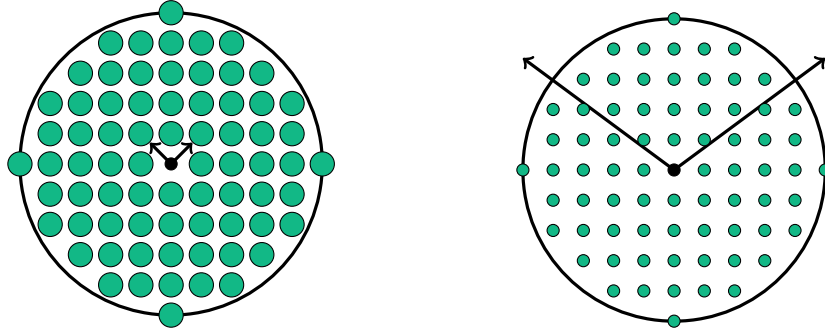
- How does the volume of K' compare to K ?
- Show that the sum of any two points in K' lies in K
- Apply Blichfeldt’s theorem to K' to prove Minkowski’s theorem.

Solution

- The volume of K' is $\frac{1}{2^n}$ the volume of K .
- Take $x, y \in K'$. It follows that $2x, 2y \in K$. Since K is convex, we have that the midpoint of the line segment between $2x$ and $2y$ is in K , and so $\frac{2x+2y}{2} = x+y \in K$.
- Since the volume of K is greater than 2^n , we have the volume of K' is greater than one. Applying Blichfeldt’s theorem, we can find two distinct points $x, y \in K'$ such that $x-y \in \mathbb{Z}^n$. Since K' is symmetric with respect to the origin, we have that $-y \in K'$. Therefore, $x+(-y) \in K$ by the previous part. $x \neq y, x-y \neq 0$, so we have found a nontrivial integer point in K .

Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius R . A tree of radius r has been planted at every integer point in the circle. If r is small, you will have a clear line of sight through the orchard. If r is large, there will be no clear line of sight through in any direction:



Problem 23:

Show that you will have at least one clear line of sight if $r < \frac{1}{\sqrt{R^2+1}}$.

Hint: Consider the line segment from $(0, 0)$ to $(R, 1)$. Calculate the distance from the closest integer points to the ray.

Solution

Consider the ray from the origin to the point $(R, 1)$.

The two lattice points closest to this ray are $(1, 0)$ and $(R-1, 1)$. Say the distance from each of these points to the ray is δ .

Now, consider the triangle with vertices $(0, 0)$, $(1, 0)$, and $(R, 1)$. The area of this triangle is $\frac{1}{2}$.

The area of this triangle is also equal to $\frac{1}{2}\delta\sqrt{R^2+1}$. By algebra, $\delta = \frac{1}{\sqrt{R^2+1}}$.

Therefore, if $r < \frac{1}{\sqrt{R^2+1}}$, we will have a clear line of sight given by this ray.

Problem 24:

Show that there is no line of sight through the orchard if $r > \frac{1}{R}$. You may want to use the following steps:

- Show that there is no line of sight if $r \geq 1$.
- Suppose $r < 1$ and $r > \frac{1}{R}$. Then, $R \geq 2$. Choose a potential line of sight passing through an arbitrary point P on the circle. Thicken this line of sight equally on both sides into a rectangle of width $2r$ tangent to P and $-P$. From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

Solution

Suppose $r < 1$ and let L be a potential line of sight. Consider the rectangle of width $2r$ tangent to P and $-P$. Then this is convex and symmetric with respect to the origin. Its area is $(2R)(2r) > 4\frac{R}{R} = 4$. By Minkowski, we have a nonzero integral point in this rectangle. Suppose first that the integer point is within the orchard. Then this means that there is a tree whose distance to the line is at most r . Therefore, this tree blocks the line of sight. Now notice that there is a part of this rectangle that sits outside the orchard. Can the integer point be in this region? This would mean its distance to the origin, D , would satisfy $D > R$. Now since this point is within a distance of r of our line L , we have that $D < \sqrt{R^2 + r^2} < \sqrt{R^2 + 1}$. So we have that $R < D < \sqrt{R^2 + 1}$. Then $R^2 < D^2 < R^2 + 1$, but D^2 is an integer so this is impossible.

Problem 25: Challenge

Prove that there exists a rational approximation of $\sqrt{3}$ within 10^{-3} with denominator at most 501. Come up with an upper bound for the smallest denominator of a ϵ -close rational approximation of any irrational number $\alpha > 0$. Your bound can have some dependence on α and should get smaller as α gets larger.

Hint: Use the orchard.

Solution

Take the line through the origin of slope $\sqrt{3}$. We would like an orchard for which $r = 10^{-3}$ gives no line of sight, since this will guarantee an integer point within a distance of 10^{-3} . Then by our previous problem, we can take $10^{-3} > \frac{1}{R}$, so take $R > 1000$. Now since this line intersects the boundary of the orchard at $(\frac{R}{2}, \frac{\sqrt{3}R}{2})$, we have that the x -coordinate is at most $\frac{R}{2} = 501$. Then we have that our lattice point (x, y) satisfies $\sqrt{3}x - 10^{-3} < y < \sqrt{3}x + 10^{-3}$, so $\sqrt{3} - 10^{-3} < \frac{y}{x} < \sqrt{3} + \frac{10^{-3}}{x}$. Therefore, $\frac{y}{x}$ is a rational approximation that is 10^{-3} -close to $\sqrt{3}$ and has denominator at most 501. Notice that we got closer than we need to. Repeating this same process, our upper bound for the denominator of an ϵ -close approximation of α is $\frac{\cos(\text{atan}(\alpha))}{\epsilon} = \frac{1}{\sqrt{\alpha^2 + 1}\epsilon}$.