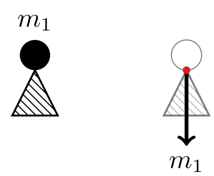
Geometry of Masses I

Prepared by Sunny & Mark on January 23, 2025

Part 1: Balancing a line

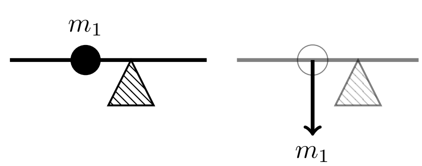
Example 1:
Consider a mass m_1 on top of a pin.
Due to gravity, the mass exerts a force on the pin at the point of contact.
For simplicity, we'll say that the magnitude of this force is equal the mass of the object— that is, m_1.



The pin exerts an opposing force on the mass at the same point, and the system thus stays still.

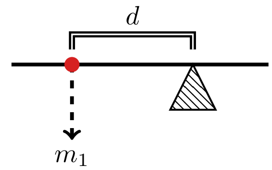
Remark 2:
Forces, distances, and torques in this handout will be provided in arbitrary (though consistent) units.
We have no need for physical units in this handout.

Example 3:
Now attach this mass to a massless rod and try to balance the resulting system.
As you might expect, it is not stable: the rod pivots and falls down.



This is because the force m_1 is offset from the pivot (i.e, the tip of the pin).
It therefore exerts a *torque* on the mass-rod system, causing it to rotate and fall.

Definition 4: Torque
Consider a rod on a single pivot point. If a force with magnitude m_1 is applied at an offset d from the pivot point, the system experiences a *torque* with magnitude $m_1 \times d$.

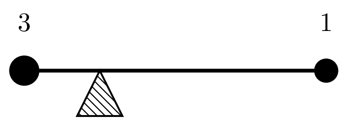


We'll say that a *positive torque* results in *clockwise* rotation, and a *negative torque* results in a *counterclockwise rotation*. As stated in **??**, torque is given in arbitrary "torque units" consistent with our units of distance and force.

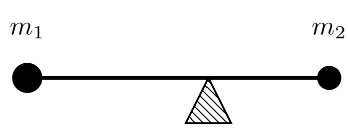
Look at the diagram above and convince yourself that this convention makes sense:
- m_1 is positive (masses are usually positive)
- d is negative (m_1 is *behind* the pivot)
- therefore, $m_1 \times d$ is negative.

Definition 5: Center of mass
The *center of mass* of a physical system is the point at which one can place a pivot so that the total torque the system experiences is 0.
In other words, it is the point at which the system may be balanced on a pin.

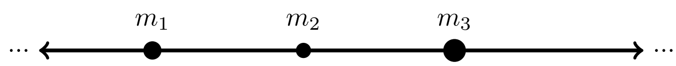
Problem 6:
Consider the following physical system: we have a massless rod of length 1, with a mass of size 3 at position 0 and a mass of size 1 at position 1. Find the position of this system's center of mass.



Problem 7:
Do the same for the following system, where m_1 and m_2 are arbitrary masses.



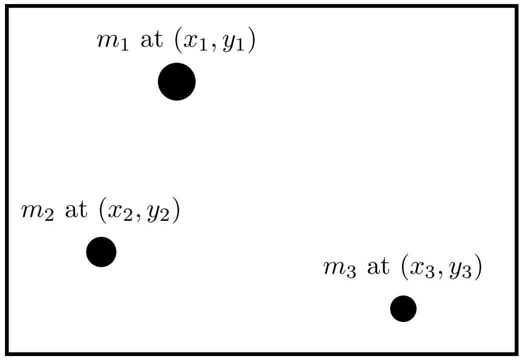
Definition 8:
Consider a massless, horizontal rod of infinite length.
Affix a finite number of point masses to this rod.
We will call the resulting object a *one-dimensional system of masses*:



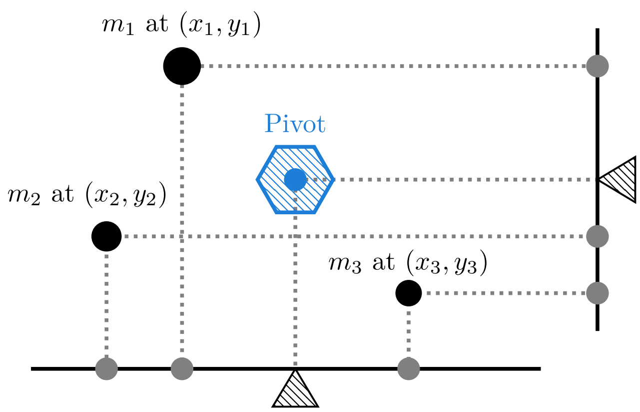
Problem 9:
Consider a one-dimensional system of masses consisting of n masses $m_1, m_2, ..., m_n$,
with each m_i positioned at x_i. Show that the resulting system always has a unique center of mass.
Hint: Prove this by construction: find the point!

Part 2: Balancing a plane

Definition 10:
Consider a massless two-dimensional plane.
Affix a finite number of point masses to this plane.
We will call the resulting object a *two-dimensional system of masses:*



Problem 11:
Show that any two-dimensional system of masses has a unique center of mass.
Hint: If a plane balances on a pin, it does not tilt in the x or y direction.
See the diagram below.



Part 3: Continuous mass

Now let's extend this idea to a *continuous distribution* of masses rather than discrete point masses. This isn't so different; a continuous distribution of mass is really just a lot of point-masses, only that there are so many of them so close together that you can't even count them[1]. In general, finding the CoM requires integral calculus, but not always...[2]

Problem 12:
You are given a cardboard cutout of a seahorse and some office supplies.
How might you determine its center of mass? Does your strategy also work in 3D?

Definition 13: Centroid
Centroids are closely related to, and often synonymous with, centers of mass. A centroid is the geometric center of an object, regardless of the mass distribution. Thus, the centroid and center of mass are the same when the mass is uniformly distributed.

Problem 14:
Where is the center of a right isosceles triangle? What about any isosceles triangle?

Problem 15:
How can you easily find the center of mass of any triangle? Why does this work?

[1]For example, your pencil might seem like a continuous distribution of mass, but it's really just a whole lot of atoms.

[2]Many of the following problems can be solved with integration even though you're meant to solve them without it. But remember, in math, whenever you accomplish the same task two different ways, that really means that they're somehow the same thing.

Problem 16:
Consider Figure **??** depicting a simplified soda can. If you leave just the right amount, you can get it to balance on the beveled edge, as seen in Figure **??**.

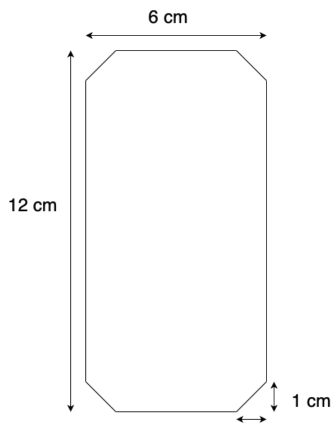


Figure 1:

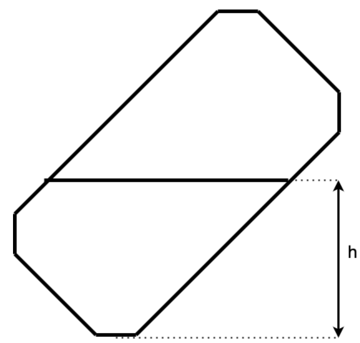


Figure 2:

Problem 17:
See Figure **??**. Let's take the can to be massless and initially empty. Let's also assume that we live in two dimensions. We start slowly filling it up with soda to a vertical height h. What is h just before the can tips over?

Problem 18:
Think about how you might approach this problem in 3D. Does h become larger or smaller?

So far we've made the assumption our shapes have mass that is *uniformly distributed*. But that doesn't have to be the case.

Problem 19:
A mathemagical wizard will give you his staff if you can balance it horizontally on your finger. The strange magical staff has unit length and it's mass is distributed in a very special way. It's density decreases linearly from λ_0 at one end to 0 at the other. Where is the staff's balancing point?

Part 4: Pappus's Centroid Theorem

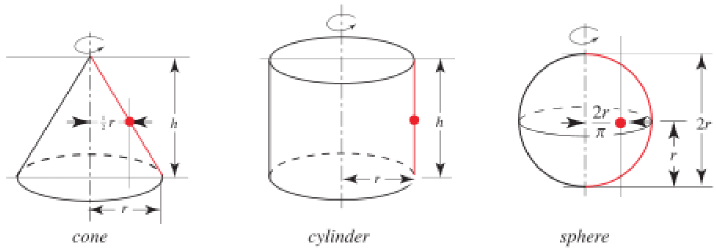


Figure 3:

Remark 20:
Centroids are closely related to, and often synonymous with, centres of mass. A centroid is the geometric centre of an object, regardless of the mass distribution. Thus, the centroid and centre of mass coincide when the mass is uniformly distributed.

Remark 21:
Figure **??** depicts three different surfaces constructed by revolving a line segment (in red) about a central axis. These are often called *surfaces of revolution*.

Problem 22:
Pappus's First Centroid Theorem allows you to determine the area of a surface of revolution using information about the line segment and the axis of rotation. Can you intuitively come up with Pappus's First Centroid Theorem for yourself? Figure **??** is very helpful. It may also help to draw from surface area formulae you already know. What limitations are there on the theorem?

Problem 23:
Pappus's Second Centroid Theorem simply extends this concept to *solids of revolution*, which are exactly what you think they are.

Problem 24:
Now that you've done the first theorem, what do you think Pappus's Second Centroid Theorem states?

Problem 25:
The centroid of a semi-circular line segment is already given in Figure **??**, but what about the centroid of a filled semi-circle? (Hint: For a sphere of radius r, $V = \frac{4}{3}\pi r^3$)

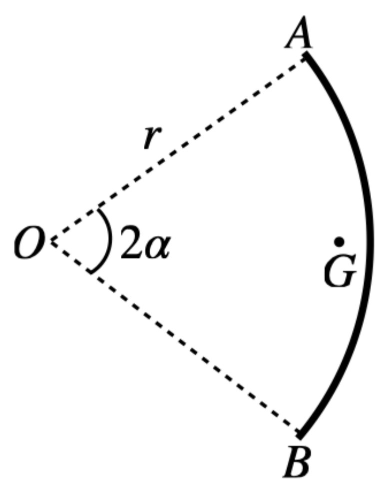


Figure 4:

Problem 26:
Given arc AB with radius r and subtended by 2α, determine OG, the distance from the centre of the circle to the centroid of the arc.

Problem 27:
Where is the centroid of the *sector* of the circle in Figure **??**? *Hint:* cut it up.

Problem 28:
Seeing your success with his linear staff, the wizard challenges you with another magical staff to balance. It looks identical to the first one, but you're told that the density decreases from λ_0 to 0 according to the function $\lambda(x) = \lambda_0\sqrt{1 - x^2}$.

Problem 29:
Infinitely many masses m_i are placed at x_i along the positive x-axis, starting with $m_0 = 1$ placed at $x_0 = 1$. Each successive mass is placed twice as far from the origin compared to the previous one. But also, each successive mass has a quarter the weight of the previous one. Find the CoM if it exists.

Problem 30: Bonus
Try to actually find h from Problem **??**. Good luck.