Tropical Polynomials

Prepared by Mark on January 23, 2025
Based on a handout by Bryant Mathews

Instructor's Handout
This handout contains solutions and notes.
Recompile without solutions before distributing.

Section 1: Tropical Arithmetic

Definition 1:
The *tropical sum* of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

Definition 2:
The *tropical product* of two numbers is their sum:

$$x \otimes y = x + y$$

Problem 3:

- Is tropical addition commutative?
  i.e, does $x \oplus y = y \oplus x$?
- Is tropical addition associative?
  i.e, does $(x \oplus y) \oplus z = x \oplus (y \oplus z)$?
- Is there a tropical additive identity?
  i.e, is there an i so that $x \oplus i = x$ for all real x?

Solution:

- Is tropical addition commutative?
  Yes, $\min(\min(x, y), z) = \min(x, y, z) = \min(x, \min(y, z))$
- Is tropical addition associative?
  Yes, $\min(x, y) = \min(y, x)$
- Is there a tropical additive identity?
  No. There is no n where $x \leq n$ for all real x

Problem 4:
Let's expand $\mathbb{R}$ to include a tropical additive identity.

- What would be an appropriate name for this new number?
- Give a reasonable definition for…
  - the tropical sum of this number and a real number x
  - the tropical sum of this number and itself
  - the tropical product of this number and a real number x
  - the tropical product of this number and itself

Solution:

∞ makes sense, with $\infty \oplus x = x$; $\infty \oplus \infty = \infty$; $\infty \otimes x = \infty$; and $\infty \otimes \infty = \infty$

Problem 5:
Do tropical additive inverses exist?
Is there an inverse y for every x so that $x \oplus y = \infty$?
Remember that ∞ is the additive identity.

Solution:

No. Unless $x = \infty$, there is no x where $\min(x, y) = \infty$

Problem 6:
Is tropical multiplication associative?
Does $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ for all x, y, z?

Solution:

Yes, since (normal) addition is associative

Problem 7:
Is tropical multiplication commutative?
Does $x \otimes y = y \otimes x$ for all x, y?

Solution:

Yes, since (normal) addition is commutative

Problem 8:
Is there a tropical multiplicative identity?
Is there an i so that $x \otimes i = x$ for all x?

Solution:

Yes, it is 0.

Problem 9:
Do tropical multiplicative inverses always exist?
For every $x \neq \infty$, does there exist an inverse y so that $x \otimes y = i$,
where i is the additive identity?

Solution:

Yes, it is $-x$. For $x \neq 0$, $x \otimes (-x) = 0$

Problem 10:
Is tropical multiplication distributive over addition?
Does $x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$?

Solution:

Yes, $x + \min(y, z) = \min(x + y, x + z)$

Problem 11:
Fill the following tropical addition and multiplication tables

Solution:

$\oplus$	1	2	3	4	∞
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
∞	1	2	3	4	∞

$\otimes$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

Problem 12:
Expand and simplify $f(x) = (x \oplus 2)(x \oplus 3)$, then evaluate $f(1)$ and $f(4)$
Hint: Adjacent parenthesis imply tropical multiplication

Solution:

$$\begin{aligned}(x \oplus 2)(x \oplus 3) &= x^2 \oplus 2x \oplus 3x \oplus (2 \otimes 3)\\ &= x^2 \oplus (2 \oplus 3)x \oplus (2 \otimes 3)\\ &= x^2 \oplus 2x \oplus 5\end{aligned}$$

Also, $f(1) = 2$ and $f(4) = 5$.

Section 2: Tropical Polynomials

Definition 13:
A *polynomial* is an expression formed by adding and multiplying numbers and a variable x.
Every polynomial can be written as

$$c_0 + c_1 x + c_2 x^2 + ... + c_n x^n$$

for some nonnegative integer n and coefficients $c_0, c_1, ..., c_n$.
The *degree* of a polynomial is the largest n for which c_n is nonzero.

Theorem 14:
The *fundamental theorem of algebra* states that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with real coefficients.

For example, the polynomial $-160 - 64x - 2x^2 + 17x^3 + 8x^4 + x^5$
can be written as $(x^2 + 2x + 5)(x - 2)(x + 4)(x + 4)$

A similar theorem exists for polynomials with complex coefficients.
These coefficients may be found using the *roots* of this polynomial.
As you already know, there are formulas that determine the roots of quadratic, cubic, and quartic (degree 2, 3, and 4) polynomials. There are no formulas for the roots of polynomials with larger degrees—in this case, we usually rely on appropriate roots found by computers.

In this section, we will analyze tropical polynomials:

- Is there a fundamental theorem of tropical algebra?
- Is there a tropical quadratic formula? How about a cubic formula?
- Is it difficult to find the roots of tropical polynomials with large degrees?

Definition 15:
A *tropical* polynomial is a polynomial that uses tropical addition and multiplication.
In other words, it is an expression of the form

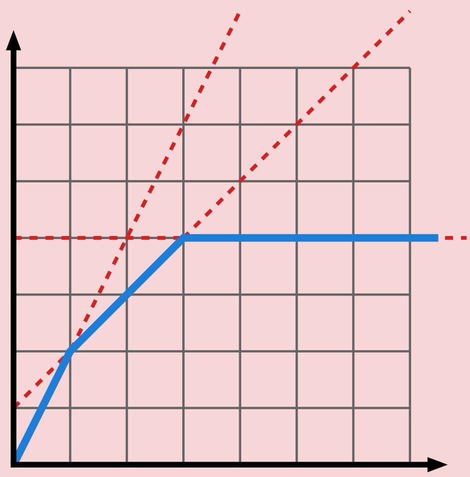
$$c_0 \oplus (c_1 \otimes x) \oplus (c_2 \otimes x^2) \oplus ... \oplus (c_n \otimes x^n)$$

where all exponents represent repeated tropical multiplication.

Problem 16:
Draw a graph of the tropical polynomial $f(x) = x^2 \oplus 1x \oplus 4$.
Hint: $1x$ is not equal to x.

Solution:

$f(x) = \min(2x, 1 + x, 4)$, which looks like:



Problem 17:
Now, factor $f(x) = x^2 \oplus 1x \oplus 4$ into two polynomials with degree 1.
In other words, find r and s so that

$$x^2 \oplus 1x \oplus 4 = (x \oplus r)(x \oplus s)$$

Naturally, we will call r and s the *roots* of f.

Solution:

Because $(x \oplus r)(x \oplus s) = x^2 \oplus (r \oplus s)x \oplus sr$, we must have $r \oplus s = 1$ and $r \otimes s = 4$.
In standard notation, we need $\min(r, s) = 1$ and $r + s = 4$, so we take $r = 1$ and $s = 3$:

$$f(x) = x^2 \oplus 1x \oplus 4 = (x \oplus 1)(x \oplus 3)$$

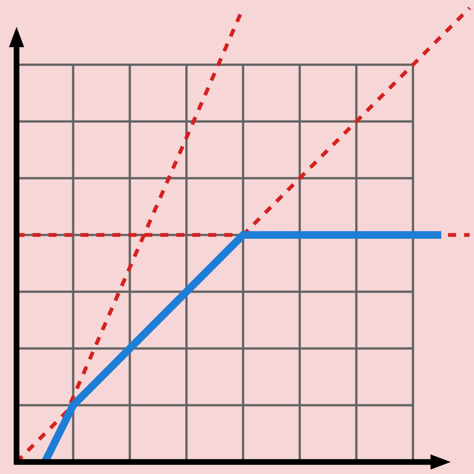
Problem 18:
Can you see the roots of this polynomial in the graph?
Hint: Yes, you can. What "features" do the roots correspond to?

Solution:

The roots are the corners of the graph.

Problem 19:
Graph $f(x) = -2x^2 \oplus x \oplus 8$.
Hint: Use half scale. 1 box = 2 units.

Solution:



Problem 20:
Find a factorization of f in the form $a(x \oplus r)(x \oplus s)$.

Solution:

We (tropically) factor out -2 to get

$$f(x) = -2(x^2 \oplus 2x \oplus 10)$$

by the same process as the previous problem, we get

$$f(x) = -2(x \oplus 2)(x \oplus 8)$$

Problem 21:
Can you see the roots r and s in the graph?
How are the roots related to the coefficients of f?
Hint: look at consecutive coefficients: $0 - (-2) = 2$

Solution:

The roots are the differences between consecutive coefficients of f:

- $0 - (-2) = 2$
- $8 - 0 = 8$

Problem 22:
Find a tropical polynomial that has roots 4 and 5
and always produces 7 for sufficiently large inputs.

Solution:

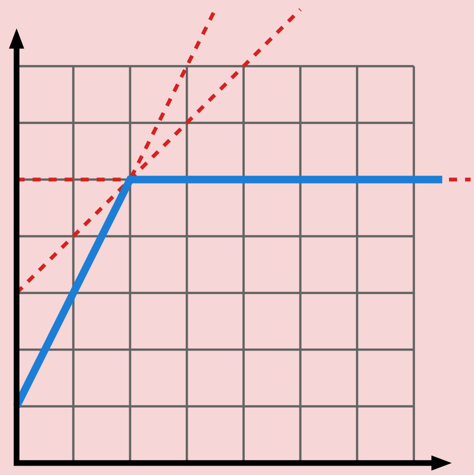
We are looking for $f(x) = ax^2 \oplus bx \oplus c$.
Since $f(\infty) = 7$, we know that $c = 7$.
Using the pattern from the previous problem, we'll subtract 5 from c to get $b = 2$,
and 4 from b to get $a = -2$.
And so, $f(x) = -2x^2 \oplus 2x \oplus 7$

Subtracting roots in the opposite order does not work.

Problem 23:
Graph $f(x) = 1x^2 \oplus 3x \oplus 5$.

Solution:

The graphs of all three terms intersect at the same point:



Problem 24:
Find a factorization of f in the form $a(x \oplus r)(x \oplus s)$.

Solution:

$$f(x) = 1x^2 \oplus 3x \oplus 5 = 1(x \oplus 2)^2$$

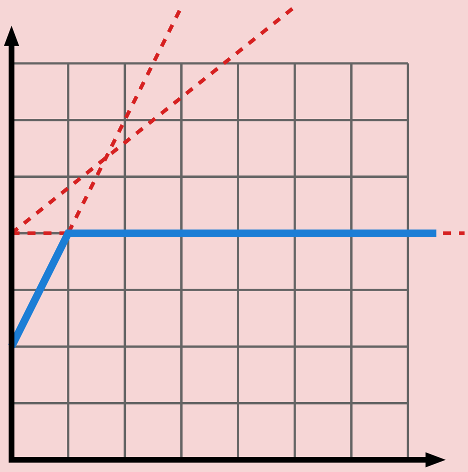
Problem 25:
How is this graph different from the previous two?
How is this polynomial's factorization different from the previous two?
How are the roots of f related to its coefficients?

Solution:

The factorization contains the same term twice.
Also note that the differences between consecutive coefficients of f are both two.

Problem 26:
Graph $f(x) = 2x^2 \oplus 4x \oplus 4$.

Solution:



Problem 27:
Find a factorization of f in the form $a(x \oplus r)(x \oplus s)$, or show that one does not exist.

Solution:

We can factor out a 2 to get $f(x) = 2(x^2 \oplus 2x \oplus 2)$, but $x^2 \oplus 2x \oplus 2$ does not factor. There are no a and b with minimum 2 and sum 2.

Problem 28:
Find a polynomial that has the same graph as f, but can be factored.

Solution:

$$2x^2 \oplus 3x \oplus 4 = 2(x \oplus 1)^2$$

Theorem 29:
The *fundamental thorem of tropical algebra* states that for every tropical polynomial f, there exists a *unique* tropical polynomial $\bar{f}$ with the same graph that can be factored into linear factors.

Whenever we say "the roots of f", we really mean "the roots of $\bar{f}$."
Also, f and $\bar{f}$ might be the same polynomial.

Problem 30:
If $f(x) = ax^2 \oplus bx \oplus c$, then $\bar{f}(x) = ax^2 \oplus Bx \oplus c$ for some B.
Find a formula for B in terms of a, b, and c.
Hint: there are two cases to consider.

Solution:

If we want to factor $a(x^2 \oplus (b-a)x \oplus (c-a))$, we need to find r and s so that

- $\min(r, s) = b - a$, and
- $r + s = c - a$

This is possible if and only if $2(b-a) \leq c - a$,
or equivalently if $b \leq (a + c) \div 2$

Case 1: If $b \leq (a + c \div)2$, then $\bar{f} = f$ and $b = B$.

Case 2: If $b > (a + c \div)2$, then

$$\begin{aligned}\bar{f}(x) &= ax^2 \oplus \left(\frac{a+c}{2}\right)x \oplus c \\ &= a\left(x \oplus \frac{c-a}{2}\right)^2\end{aligned}$$

has the same graph as f, and thus $B = (a + c) \div 2$

We can combine these results as follows:

$$B = \min\left(b, \frac{a+c}{2}\right)$$

Problem 31:
Find a tropical quadratic formula in terms of a, b, and c
for the roots x of a tropical polynomial $f(x) = ax^2 \oplus bx \oplus c$.
Hint: again, there are two cases.
Remember that "roots of f" means "roots of $\bar{f}$".

Solution:

Case 1: If $b \leq (a + c) \div 2$, then $\bar{f} = f$ has roots $b - a$ and $c - b$, so

$$\bar{f}(x) = a(x \oplus (b - a))(x \oplus (c - b))$$

Case 2: If $b > (a + c) \div 2$, then $\bar{f}$ has root $(c - a) \div$ with multiplicity 2, so

$$\bar{f}(x) = a\left(x \oplus \frac{c-a}{2}\right)^2$$

It is interesting to note that the condition $2b < a + c$ for there to be two distinct roots becomes $b^2 > ac$ in tropical notation. This is reminiscent of the discriminant condition for standard polynomials!

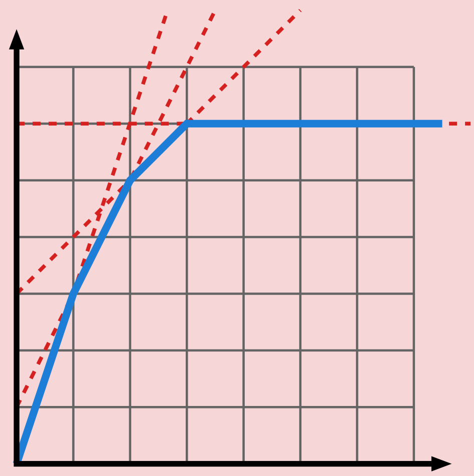
Section 3: Tropical Cubic Polynomials

Problem 32:
Consider the polynomial $f(x) = x^3 \oplus x^2 \oplus 3x \oplus 6$.

- sketch a graph of this polynomial
- use this graph to find the roots of f
- write (and expand) a product of linear factors with the same graph as f.

Solution:

- Roots are 1, 2, and 3.
- $\bar{f}(x) = x^3 \oplus 1x^2 \oplus 3x \oplus 6 = (x \oplus 1)(x \oplus 2)(x \oplus 3)$

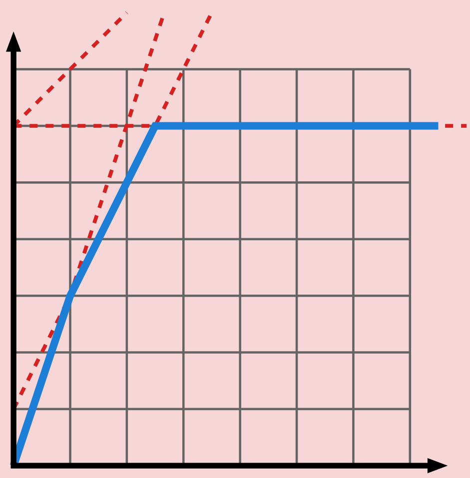


Problem 33:
Consider the polynomial $f(x) = x^3 \oplus x^2 \oplus 6x \oplus 6$.

- sketch a graph of this polynomial
- use this graph to find the roots of f
- write (and expand) a product of linear factors with the same graph as f.

Solution:

- Roots are 1, 2.5, and 2.5.
- $\bar{f}(x) = x^3 \oplus 1x^2 \oplus 3.5x \oplus 6 = (x \oplus 1)(x \oplus 2.5)^2$

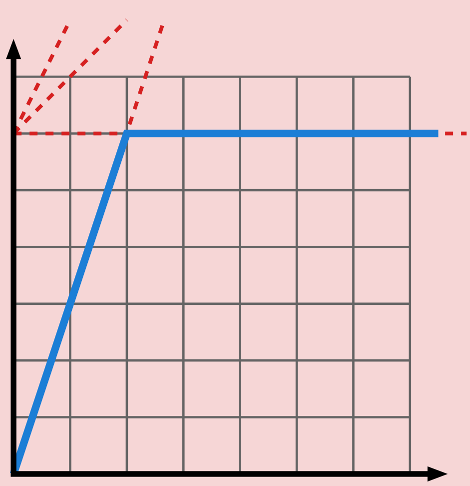


Problem 34:
Consider the polynomial $f(x) = x^3 \oplus 6x^2 \oplus 6x \oplus 6$.

- sketch a graph of this polynomial
- use this graph to find the roots of f
- write (and expand) a product of linear factors with the same graph as f.

Solution:

- Roots are 2, 2, and 2.
- $\bar{f}(x) = x^3 \oplus 2x^2 \oplus 4x \oplus 6 = (x \oplus 2)^3$



Problem 35:
If $f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d$, then $\bar{f}(x) = ax^3 \oplus Bx^2 \oplus Cx \oplus d$ for some B and C. Using the last three problems, find formulas for B and C in terms of a, b, c, and d.

Solution:

$$B = \min\left(b, \frac{a+c}{2}, \frac{2a+d}{2}\right)$$

$$C = \min\left(c, \frac{b+d}{2}, \frac{a+2d}{2}\right)$$

Problem 36:
What are the roots of the following polynomial?

$$3x^6 \oplus 4x^5 \oplus 2x^4 \oplus x^3 \oplus x^2 \oplus 4x \oplus 5$$

Solution:

We have

$$\bar{f}(x) = 3x^6 \oplus 2x^5 \oplus 1x^4 \oplus x^3 \oplus 1x^2 \oplus 3x \oplus 5$$

which has roots -1, -1, -1, 1, 2, 2

Problem 37:
If

$$f(x) = c_0 \oplus c_1 x \oplus c_2 x^2 \oplus \ldots \oplus c_n x^n$$

then

$$\bar{f}(x) = c_0 \oplus C_1 x \oplus C_2 x^2 \oplus \ldots \oplus C_{n-1} x^{n-1} \oplus c_n x^n$$

Find a formula for each C_i in terms of $c_0, c_1, \ldots, c_n$.

Solution:

$$\begin{aligned} A_j &= \min_{l \leq j < k} \left(\frac{a_l - a_k}{k - l}(k - j) + a_k \right) \\ &= \min_{l \leq j < k} \left(a_l \frac{k - j}{k - l} + a_k \frac{j - l}{k - l} \right) \end{aligned}$$

which is a weighted average of some a_l and a_k, with $l \leq j < k$

Problem 38:
With the same setup as the previous problem,
find formulas for the roots $r_1, r_2, \ldots, r_n$.

Solution:

The roots are the differences between consecutive coefficients of $\bar{f}$:

$$r_i = A_i - A_{i-1}$$

where we set $A_n = a_n$ and $A_0 = a_0$.

Problem 39:
Can you find a geometric interpretation of these formulas
in terms of the points $(-i, c_i)$ for $0 \leq i \leq n$?

Solution:

The inequality (for $l \leq k < k$)

$$A_j \leq \frac{a_l - a_k}{k - l}(k - j) + a_k$$

states that the point $\left(-j, A_j\right)$ must lie on or below the line segment between the points $(-k, a_k)$ and $(-l, a_l)$. This makes it easy to find the A_j using a graph of the points $(-i, a_i)$ for $0 \leq i \leq n$.