Definable Sets

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Part 1: Logical Algebra

Definition 1:
Logical operators operate on the values $\{\texttt{true}, \texttt{false}\}$,
just like algebraic operators operate on numbers.
In this handout, we'll use the following operators:

- $\neg$: not
- $\wedge$: and
- $\vee$: or
- $\rightarrow$: implies
- $()$: parenthesis.

The function of these is defined by *truth tables*:

and

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

or

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

implies

A	B	$A \rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

not

A	$\neg A$
T	F
F	T

$A \wedge B$ is `true` only if both A and B are `true`. $A \vee B$ is `true` if A or B (or both) are `true`.
$\neg A$ is the opposite of A, which is why it looks like a "negative" sign.

$A \rightarrow B$ is a bit harder to understand. Read aloud, this is "A implies B."
The only time $\rightarrow$ produces `false` is when `true` $\rightarrow$ `false`. This fact may seem counterintuitive, but will make more sense as we progress through this handout.
Hint: Think about it—if event α implies β, it is impossible for α to occur without β.
This is the only impossibility. All other variants are valid.

Problem 2:
Evaluate the following.

- $\neg\texttt{T}$
- $\texttt{F} \vee \texttt{T}$
- $\texttt{T} \wedge \texttt{T}$
- $(\texttt{T} \wedge \texttt{F}) \vee \texttt{T}$
- $(\neg(\texttt{F} \vee \neg\texttt{T})) \rightarrow \neg\texttt{T}$
- $(\texttt{F} \rightarrow \texttt{T}) \rightarrow (\neg\texttt{F} \vee \neg\texttt{T})$

Problem 3:
Evaluate the following.

- $A \to \mathtt{T}$ for any A
- $(\neg(A \to B)) \to A$ for any A, B
- $(A \to B) \to (\neg B \to \neg A)$ for any A, B

Problem 4:
Show that $\neg(A \to \neg B)$ is equivalent to $A \land B$.
That is, show that these expressions always evaluate to the same value given the same A and B.
Hint: Use a truth table

Problem 5:
Write an expression equivalent to $A \lor B$ using only $\neg$, $\to$, and $()$?

Note that both $\land$ and $\lor$ can be defined using the other logical symbols.
The only logical symbols we *need* are $\neg$, $\to$, and $()$.
We include $\land$ and $\lor$ to simplify our expressions.

Part 2: Structures

Definition 6:
A *universe* is a set of meaningless objects. Here are a few examples:
- $\{a, b, ..., z\}$
- $\{0, 1\}$
- $\mathbb{Z}$, $\mathbb{R}$, etc.

Definition 7:
A *structure* consists of a universe and a set of *symbols*.
A structure's symbols give meaning to the objects in its universe.

Symbols come in three types:
- *Constant symbols*, which let us specify specific elements of our universe.
  Examples: $0, 1, \frac{1}{2}, \pi$
- *Function symbols*, which let us navigate between elements of our universe.
  Examples: $+, \times, \sin x, \sqrt{x}$
  Note that symbols we usually call "operators" are functions under this definition.
  The only difference between $a + b$ and $+(a, b)$ is notation.
- *Relation symbols*, which let us compare elements of our universe.
  Examples: $<, >, \leq, \geq$

The equality check $=$ is *not* a relation symbol. It is included in every structure by default.
By definition, $a = b$ is true if and only if a and b are the same element of our universe.

Example 8:
The first structure we'll look at is the following:

$$\left(\mathbb{Z} \mid \{0, 1, +, -, <\}\right)$$

This is a structure over the universe $\mathbb{Z}$ that provides the following symbols:
- Constants: $\{0, 1\}$
- Functions: $\{+, -\}$
- Relations: $\{<\}$

If we look at our set of constant symbols, we see that the only integers we can directly refer to in this structure are 0 and 1. If we want any others, we must define them using the tools this structure offers.

To "define" an element of a set, we need to write a sentence that is only true for that element.
If we want to define 2 in the structure above, we could use the following sentence:

"2 is the x that satisfies $[1 + 1 = x]$."

This is a valid definition because 2 is the *only* element of $\mathbb{Z}$ for which $[1 + 1 = x]$ evaluates to `true`.

Problem 9:
Define -1 in $\left(\mathbb{Z} \mid \{0, 1, +, -, <\}\right)$.

Let us formalize what we found in the previous two problems.

Definition 10: Formulas
A *formula* in a structure S is a well-formed string of constants, functions, relations, and logical operators.

You already know what a "well-formed string" is: $1 + 1$ is fine, $\sqrt{+}$ is nonsense.
For the sake of time, I will not provide a formal definition — it isn't particularly interesting.

As a quick example, the formula $\psi \coloneqq [\neg(1 = 1)]$ is always false,
and $\varphi(x) \coloneqq [1 + 1 = x]$ evaluates to `true` only when x is 2.

Definition 11: Free Variables
A formula can contain one or more *free variables.* These are denoted $\varphi(a, b, ...)$.
Formulas with free variables let us define "properties" that certain objects have.

For example, consider the two formulas from the previous definition, ψ and φ:

- $\psi \coloneqq [\neg(1 = 1)]$
  There are no free variables in this formula.
  In any structure, ψ is always either `true` or `false`.
- $\varphi(x) \coloneqq [1 + 1 = x]$
  This formula has one free variable, labeled x.
  The value of $\varphi(x)$ depends on the x we're talking about:
  $\varphi(72)$ is false, and $\varphi(2)$ is true.

This "free variable" notation is very similar to the function notation we are used to:
The values of both $\varphi(x) \coloneqq [x > 0]$ and $f(x) = x + 1$ depend on x.

Definition 12: Definable Elements
Let S be a structure over a universe U.
We say an element $x \in U$ is *definable in S* if we can write a formula $\varphi(x)$ that only x satisfies.

Problem 13:
Define 2 in the structure $\left(\mathbb{Z}^+ \mid \{4, \times\}\right)$.
Hint: $\mathbb{Z}^+ = \{1, 2, 3, ...\}$. Also, $2 \times 2 = 4$.

Problem 14:
Try to define 2 in the structure $\left(\mathbb{Z} \mid \{4, \times\}\right)$.
Why can't you do it?

Problem 15:
Consider the structure $\left(\mathbb{R}_0^+ \mid \{1, 2, \div\}\right)$

- Define 2^2
- Define 2^n for all positive integers n
- Define 2^{-n} for all positive integers n
- What other numbers can we define in this structure?
  Hint: There is at least one more "class" of numbers we can define.

Part 3: Quantifiers

Recall the logical symbols we introduced earlier: $(), \wedge, \vee, \neg, \to$
We will now add two more: $\forall$ (for all) and $\exists$ (exists).

Definition 16:
$\forall$ and $\exists$ are *quantifiers*. They allow us to make statements about arbitrary symbols.
Quantifiers are aptly named: they tell us *how many* symbols satisfy a certain sentence.

Let's look at $\forall$ first. If $\varphi(x)$ is a formula,
the formula $\forall x \; \varphi(x)$ is true only if φ is true for all x in our universe.

For example, take the formula $\forall x \; (0 < x)$.
In English, this means "For any x, x is bigger than zero," or simply "Any x is positive."

$\exists$ is very similar: the formula $\exists x \; \varphi(x)$ is true if there is at least one x for which $\varphi(x)$ is true.
For example, $\exists \; (0 < x)$ means "there is a positive number in our set."

Problem 17:
Which of the following are true in $\mathbb{Z}$? Which are true in $\mathbb{R}_0^+$?
$\mathbb{R}_0^+$ is the set of positive real numbers and zero.

- $\forall x \; (x \geq 0)$
- $\neg(\exists x \; (x = 0))$
- $\forall x \; [\exists y \; (y \times y = x)]$
- $\forall xy \; \exists z \; (x < z < y)$ — This is a compact way to write $\forall x \; (\forall y \; (\exists z \; (x < z < y)))$
- $\neg \exists x \; (\forall y \; (x < y))$

Problem 18:
Does the order of $\forall$ and $\exists$ in a formula matter?
What's the difference between $\exists x \; \forall y \; (x \leq y)$ and $\forall y \; \exists x \; (x \leq y)$?
Hint: Consider $\mathbb{R}^+$, the set of positive reals. Zero is not positive.
Which of the above formulas is true in $\mathbb{R}^+$, and which is false?

Problem 19:
Define 0 in $\left(\mathbb{Z} \mid \{\times\}\right)$

Problem 20:
Define 1 in $\left(\mathbb{Z} \mid \{\times\}\right)$

Problem 21:
Define -1 in $\left(\mathbb{Z} \mid \{0, <\}\right)$

Problem 22:
Let $\varphi(x)$ be a formula.
Write a formula equivalent to $\forall x\ \varphi(x)$ using only logical symbols and $\exists$.

Part 4: Definable Sets

Armed with $(), \wedge, \vee, \neg, \rightarrow, \forall$, and $\exists$, we have the tools to define sets.

Definition 23: Set-Builder Notation
Say we have a sentence $\varphi(x)$.
The set of all elements that satisfy that sentence may be written as follows:

$$\{x \mid \varphi(x)\}$$

This is read "The set of x where φ is true" or "The set of x that satisfy φ."

For example, take the formula $\varphi(x) = \exists y\ (y + y = x)$.
The set of all even integers can then be written as

$$\{x \mid \exists y\ (y + y = x)\}$$

Definition 24: Definable Sets
Let S be a structure with a universe U.
We say a subset M of U is *definable* if we can write a formula
that is true for some x if and only if M contains x.

For example, consider the structure $\big(\mathbb{Z} \mid \{+\}\big)$.
Only even numbers satisfy the formula $\varphi(x) \coloneqq \big[\exists y\ (y + y = x)\big]$,
so we can define "the set of even numbers" as $\{x \mid \exists y\ (y + y = x)\}$.
Remember—we can only use symbols that are available in our structure!

Problem 25:
The empty set is definable in any structure. How?

Problem 26:
Define $\{0, 1\}$ in $\left(\mathbb{Z}_0^+ \mid \{<\}\right)$ *Hint:* Define 0 and 1 as elements first, and remember that we can use logical symbols.

Problem 27:
Define the set of prime numbers in $\left(\mathbb{Z} \mid \{\times, \div, <\}\right)$.
Hint: A prime number is an integer that is positive and is only divisible by 1 and itself.

Problem 28:
Define $\mathbb{R}_0^+$ in $\left(\mathbb{R} \mid \{\times\}\right)$

Problem 29:
Let $\triangle$ be a relational symbol. $a \triangle b$ is only true if a divides b.
Define the set of prime numbers in $\left(\mathbb{Z}^+ \mid \{\triangle\}\right)$

Theorem 30: Lagrange's Four Square Theorem
Every natural number may be written as a sum of four integer squares.

Problem 31:
Define $\mathbb{Z}_0^+$ in $\left(\mathbb{Z} \mid \{\times, +\}\right)$

Problem 32:
Define $<$ in $\left(\mathbb{Z} \mid \{\times, +\}\right)$
Hint: We can't formally define a relation yet. Don't worry about that for now.
You can repharase this question as "given $x, y \in \mathbb{Z}$, write a formula $\varphi(x, y)$ that is only true if $x < y$"

Problem 33:
Consider the structure $S = (\mathbb{R} \mid \{0, \diamond\})$
The relation $a \diamond b$ holds if $|a - b| = 1$

Part 1:
Define $\{-1, 1\}$ in S.

Part 2:
Define $\{-2, 2\}$ in S.

Problem 34:
Let $\mathcal{P}$ be the set of all subsets of $\mathbb{Z}_0^+$. This is called the *power set* of $\mathbb{Z}_0^+$.
Let S be the structure $(\mathcal{P} \mid \{\subseteq\})$

Part 1:
Show that the empty set is definable in S.
Hint: Defining $\{\}$ with $\{x \mid \neg x = x\}$ is **not** what we need here.
We need $\varnothing \in \mathcal{P}$, the "empty set" element in the power set of $\mathbb{Z}_0^+$.

Part 2:
Let $x \Bumpeq y$ be a relation on $\mathcal{P}$. $x \Bumpeq y$ holds if $x \cap y \neq \{\}$.
Show that $\Bumpeq$ is definable in S.

Part 3:
Let f be the function on $\mathcal{P}$ defined by $f(x) = \mathbb{Z}_0^+ - x$. This is called the *complement* of x.
Show that f is definable in S.
Hint: You can define a function by writing a formula $\varphi(x, y)$ that is only true when $y = f(x)$.

Part 5: Equivalence

Notation:
Let S be a structure and φ a formula.
If φ is true in S, we write $S \models \varphi$.
This is read "S satisfies φ"

Definition 35:
Let S and T be structures.
We say S and T are *equivalent* (and write $S \equiv T$) if for any formula φ, $S \models \varphi \Longleftrightarrow T \models \varphi$.
If S and T are not equivalent, we write $S \not\equiv T$.

Problem 36:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{R} \mid \{+, 0\}\Big)$

Problem 37:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{N} \mid \{+, 0\}\Big)$

Problem 38:
Show that $\Big(\mathbb{R} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{N} \mid \{+, 0\}\Big)$

Problem 39:
Show that $\Big(\mathbb{R} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{Z}^2 \mid \{+, 0\}\Big)$

Problem 40:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{Z}^2 \mid \{+, 0\}\Big)$