# Tropical Polynomials

Prepared by Mark on January 25, 2025
Based on a handout by Bryant Mathews

## Section 1: Tropical Arithmetic

**Definition 1:**
The *tropical sum* of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

**Definition 2:**
The *tropical product* of two numbers is their sum:

$$x \otimes y = x + y$$

**Problem 3:**

- Is tropical addition commutative?
  i.e, does $x \oplus y = y \oplus x$?
- Is tropical addition associative?
  i.e, does $(x \oplus y) \oplus z = x \oplus (y \oplus z)$?
- Is there a tropical additive identity?
  i.e, is there an $i$ so that $x \oplus i = x$ for all real $x$?

**Problem 4:**
Let's expand $\mathbb{R}$ to include a tropical additive identity.

- What would be an appropriate name for this new number?
- Give a reasonable definition for...
  - the tropical sum of this number and a real number $x$
  - the tropical sum of this number and itself
  - the tropical product of this number and a real number $x$
  - the tropical product of this number and itself

**Problem 5:**
Do tropical additive inverses exist?
Is there an inverse $y$ for every $x$ so that $x \oplus y = \infty$?

**Problem 6:**
Is tropical multiplication associative?
Does $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ for all $x, y, z$?

**Problem 7:**
Is tropical multiplication commutative?
Does $x \otimes y = y \otimes x$ for all $x, y$?

**Problem 8:**
Is there a tropical multiplicative identity?
Is there an $i$ so that $x \otimes i = x$ for all $x$?

**Problem 9:**
Do tropical multiplicative inverses always exist?
For every $x \neq \infty$, does there exist an inverse $y$ so that $x \otimes y = i$,
where $i$ is the additive identity?

**Problem 10:**
Is tropical multiplication distributive over addition?
Does $x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$?

**Problem 11:**
Fill the following tropical addition and multiplication tables

| $\oplus$ | 1 | 2 | 3 | 4 | $\infty$ |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| $\infty$ | | | | | |

| $\otimes$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | | | | | |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |

**Problem 12:**
Expand and simplify $f(x) = (x \oplus 2)(x \oplus 3)$, then evaluate $f(1)$ and $f(4)$
Adjacent parenthesis imply tropical multiplication

# Section 2: Tropical Polynomials

**Definition 13:**
A *polynomial* is an expression formed by adding and multiplying numbers and a variable $x$.
Every polynomial can be written as

$$c_0 + c_1 x + c_2 x^2 + ... + c_n x^n$$

for some nonnegative integer $n$ and coefficients $c_0, c_1, ..., c_n$.
The *degree* of a polynomial is the largest $n$ for which $c_n$ is nonzero.

**Theorem 14:**
The *fundamental theorem of algebra* implies that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with real coefficients.

For example, the polynomial $-160 - 64x - 2x^2 + 17x^3 + 8x^4 + x^5$
can be written as $(x^2 + 2x + 5)(x - 2)(x + 4)(x + 4)$

A similar theorem exists for polynomials with complex coefficients.
These coefficients may be found using the roots of this polynomial.
As it turns out, there are formulas that determine the roots of quadratic, cubic, and quartic (degree 2, 3, and 4) polynomials. There are no formulas for the roots of polynomials with larger degrees—in this case, we usually rely on approximate roots found by computers.

In this section, we will analyze tropical polynomials:

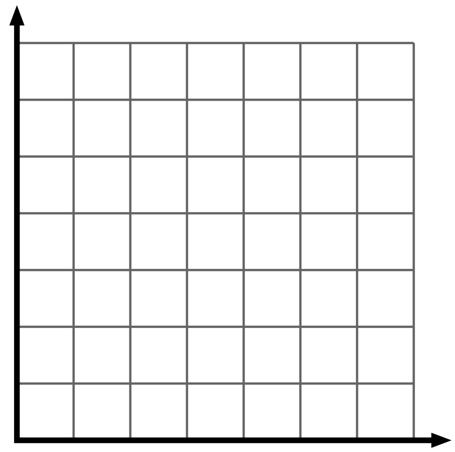
- Is there a fundamental theorem of tropical algebra?
- Is there a tropical quadratic formula? How about a cubic formula?
- Is it difficult to find the roots of tropical polynomials with large degrees?

**Definition 15:**
A *tropical* polynomial is a polynomial that uses tropical addition and multiplication.
In other words, it is an expression of the form

$$c_0 \oplus (c_1 \otimes x) \oplus (c_2 \otimes x^2) \oplus ... \oplus (c_n \otimes x^n)$$

where all exponents represent repeated tropical multiplication.

**Problem 16:**
Draw a graph of the tropical polynomial $f(x) = x^2 \oplus 1x \oplus 4$.
*Hint:* $1x$ is not equal to $x$.



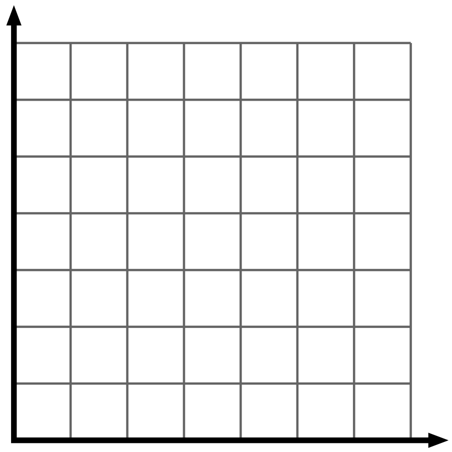
**Problem 17:**
Now, factor $f(x) = x^2 \oplus 1x \oplus 4$ into two polynomials with degree 1.
In other words, find $r$ and $s$ so that

$$x^2 \oplus 1x \oplus 4 = (x \oplus r)(x \oplus s)$$

we will call $r$ and $s$ the *roots* of $f$.

**Problem 18:**
How can we use the graph to determine these roots?

**Problem 19:**
Graph $f(x) = -2x^2 \oplus x \oplus 8$.
*Hint:* Use half scale. 1 box = 2 units.

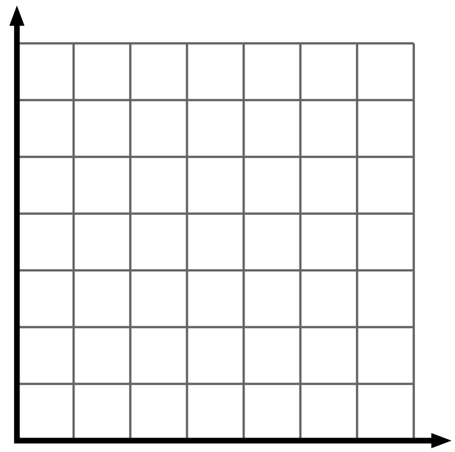


**Problem 20:**
Find a factorization of $f$ in the form $a(x \oplus r)(x \oplus s)$.

**Problem 21:**
Can you see the roots $r$ and $s$ in the graph?
How are the roots related to the coefficients of $f$?
*Hint:* look at consecutive coefficients: $0 - (-2) = 2$

**Problem 22:**
Find a tropical polynomial that has roots 4 and 5
and always produces 7 for sufficiently large inputs.

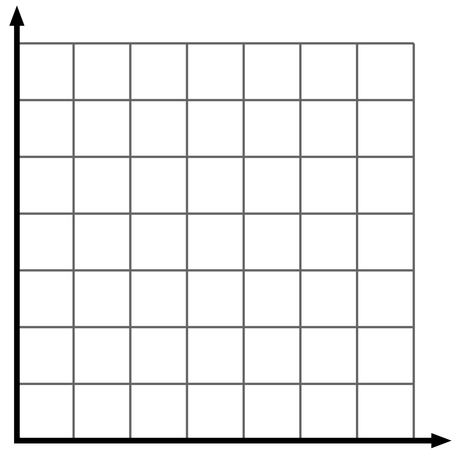
**Problem 23:**
Graph $f(x) = 1x^2 \oplus 3x \oplus 5$.



**Problem 24:**
Find a factorization of $f$ in the form $a(x \oplus r)(x \oplus s)$.

**Problem 25:**
How is this graph different from the previous two?
How is this polynomial's factorization different from the previous two?
How are the roots of $f$ related to its coefficients?

**Problem 26:**
Graph $f(x) = 2x^2 \oplus 4x \oplus 4$.



**Problem 27:**
Find a factorization of $f$ in the form $a(x \oplus r)(x \oplus s)$, or show that one does not exist.

**Problem 28:**
Find a polynomial that has the same graph as $f$, but can be factored.

**Theorem 29:**
The *fundamental thorem of tropical algebra* states that for every tropical polynomial $f$, there exists a *unique* tropical polynomial $\bar{f}$ with the same graph that can be factored into linear factors.

Whenever we say "the roots of $f$", we really mean "the roots of $\bar{f}$."
$f$ and $\bar{f}$ might be the same polynomial.

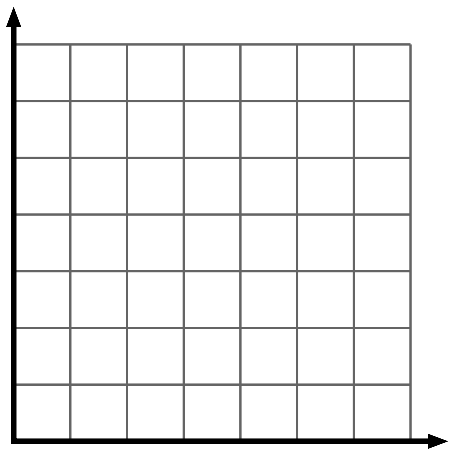
**Problem 30:**
If $f(x) = ax^2 \oplus bx \oplus c$, then $\bar{f}(x) = ax^2 \oplus Bx \oplus c$ for some $B$.
Find a formula for $B$ in terms of $a$, $b$, and $c$.
*Hint:* there are two cases to consider.

**Problem 31:**
Find a tropical quadratic formula in terms of $a$, $b$, and $c$
for the roots $x$ of a tropical polynomial $f(x) = ax^2 \oplus bx \oplus c$.
*Hint:* again, there are two cases.
Remember that "roots of $f$" means "roots of $\bar{f}$".

# Section 3: Tropical Cubic Polynomials

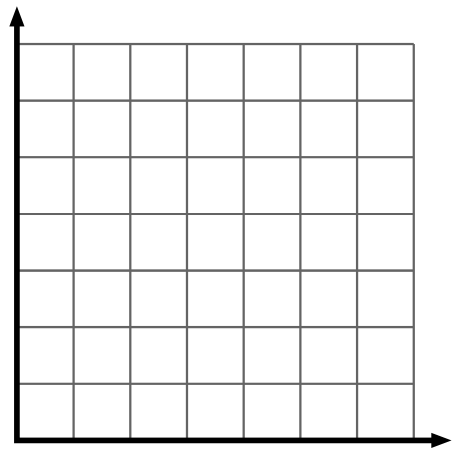
**Problem 32:**
Consider the polynomial $f(x) = x^3 \oplus x^2 \oplus 3x \oplus 6$.

- sketch a graph of this polynomial
- use this graph to find the roots of $f$
- write (and expand) a product of linear factors with the same graph as $f$.



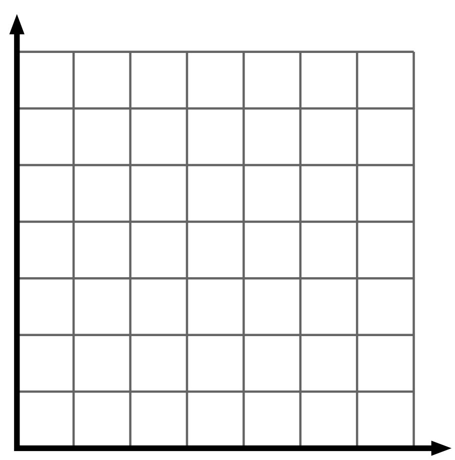
**Problem 33:**
Consider the polynomial $f(x) = x^3 \oplus x^2 \oplus 6x \oplus 6$.

- sketch a graph of this polynomial
- use this graph to find the roots of $f$
- write (and expand) a product of linear factors with the same graph as $f$.



**Problem 34:**
Consider the polynomial $f(x) = x^3 \oplus 6x^2 \oplus 6x \oplus 6$.

- sketch a graph of this polynomial
- use this graph to find the roots of $f$
- write (and expand) a product of linear factors with the same graph as $f$.



**Problem 35:**

If $f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d$, then $\bar{f}(x) = ax^3 \oplus Bx^2 \oplus Cx \oplus d$ for some $B$ and $C$. Using the last three problems, find formulas for $B$ and $C$ in terms of $a$, $b$, $c$, and $d$.

**Problem 36:**
What are the roots of the following polynomial?

$$3x^6 \oplus 4x^5 \oplus 2x^4 \oplus x^3 \oplus x^2 \oplus 4x \oplus 5$$

**Problem 37:**
If

$$f(x) = c_0 \oplus c_1 x \oplus c_2 x^2 \oplus ... \oplus c_n x^n$$

then

$$\bar{f}(x) = c_0 \oplus C_1 x \oplus C_2 x^2 \oplus ... \oplus C_{n-1} x^{n-1} \oplus c_n x^n$$

Find a formula for each $C_i$ in terms of $c_0, c_1, ..., c_n$.

**Problem 38:**
With the same setup as Problem 37,
find formulas for the roots $r_1, r_2, ..., r_n$.

**Problem 39:**
Can you find a geometric interpretation of these formulas
in terms of the points $(-i, c_i)$ for $0 \leq i \leq n$?