# Lattices

Prepared by Mark on January 25, 2025

**Definition 1:**
The *integer lattice* $\mathbb{Z}^n \subset \mathbb{R}^n$ is the set of points with integer coordinates.

**Problem 2:**
Draw $\mathbb{Z}^2$.

**Definition 3:**
We say a set of vectors $\{v_1, v_2, ..., v_k\}$ *generates* $\mathbb{Z}^n$ if every lattice point can be written uniquely as

$$a_1 v_1 + a_2 v_2 + ... + a_k v_k$$

for integer coefficients $a_i$.
It is fairly easy to show that $k$ must be at least $n$.

**Problem 4:**
Which of the following generate $\mathbb{Z}^2$?
- $\{(1,2),(2,1)\}$
- $\{(1,0),(0,2)\}$
- $\{(1,1),(1,0),(0,1)\}$

**Problem 5:**
Find a set of two vectors that generates $\mathbb{Z}^2$.
Don't say $\{(0,1),(1,0)\}$, that's too easy.

**Problem 6:**
Find a set of vectors that generates $\mathbb{Z}^n$.

**Definition 7:**
A *fundamental region* of a lattice is the parallelepiped spanned by a generating set. The exact shape of this region depends on the generating set we use.

**Problem 8:**
Draw two fundamental regions of $\mathbb{Z}^2$ using two different generating sets. Verify that their volumes are the same.

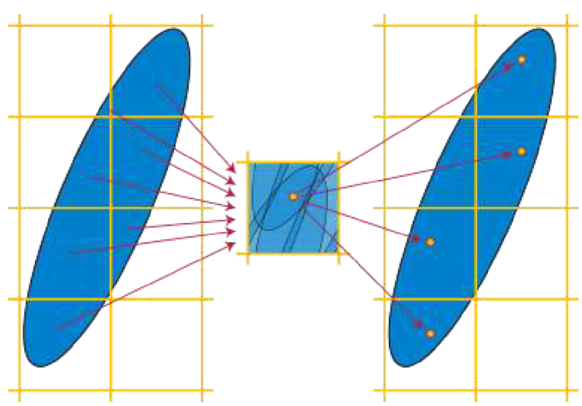
# Part 1: Minkowski's Theorem

**Theorem 9: Blichfeldt's Theorem**
Let $X$ be a finite connected region. If the volume of $X$ is greater than 1, $X$ must contain two distinct points that differ by an element of $\mathbb{Z}^n$. In other words, there exist distinct $x, y \in X$ so that $x - y \in \mathbb{Z}^n$.

Intuitively, this means that you can translate $X$ to cover two lattice points at the same time.

**Problem 10:**
Draw a region in $\mathbb{R}^2$ with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint:* Area is two-dimensional volume.

**Problem 11:**
The following picture gives an idea for the proof of Blichfeldt's theorem in $\mathbb{Z}^2$. Explain the picture and complete the proof.



**Problem 12:**
Let $X$ be a region $\in \mathbb{R}^2$ of volume $k$. How many integral points must $X$ contain after a translation?

**Definition 13:**
A region $X$ is *convex* if the line segment connecting any two points in $X$ lies entirely in $X$.

**Problem 14:**

- Draw a convex region in the plane.
- Draw a region that is not convex.

**Definition 15:**
We say a region $X$ is *symmetric* if for all points $x \in X$, $-x$ is also in $X$.

**Problem 16:**
- Draw a symmetric region.
- Draw an asymmetric region.

**Theorem 17: Minkowski's Theorem**
Every convex set in $\mathbb{R}^n$ that is symmetric with respect to the origin and which has a volume greater than $2^n$ contains an integral point that isn't zero.

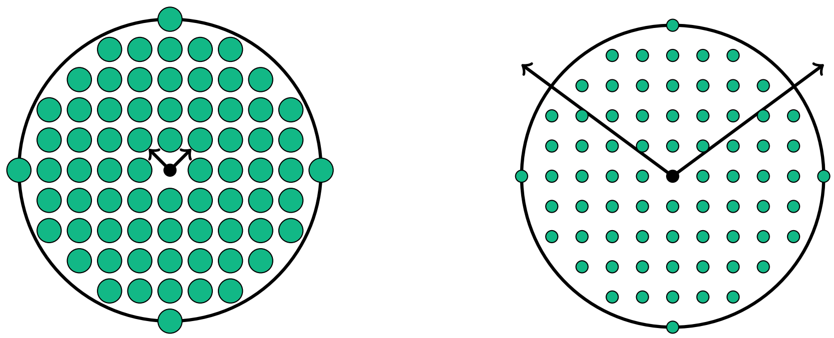
**Problem 18:**
Draw a few sets that satisfy **??** in $\mathbb{R}^2$.
What is the simplest region that has the properties listed above?

**Problem 19:**
Let $K$ be a region in $\mathbb{R}^2$ satisfying **??**.
Let $K'$ be this region scaled by $\frac{1}{2}$.
- How does the volume of $K'$ compare to $K$?
- Show that the sum of any two points in $K'$ lies in $K$ *Hint:* Use convexity.
- Apply Blichfeldt's theorem to $K'$ to prove Minkowski's theorem in $\mathbb{R}^2$.

**Problem 20:**
Let $K$ be a region in $\mathbb{R}^n$ satisfying **??**. Scale this region by $\frac{1}{2}$, called $K' = \frac{1}{2}K$.
- How does the volume of $K'$ compare to $K$?
- Show that the sum of any two points in $K'$ lies in $K$
- Apply Blichfeldt's theorem to $K'$ to prove Minkowski's theorem.

## Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius $R$. A tree of radius $r$ has been planted at every integer point in the circle. If $r$ is small, you will have a clear line of sight through the orchard. If $r$ is large, there will be no clear line of sight through in any direction:



**Problem 21:**
Show that you will have at least one clear line of sight if $r < \frac{1}{\sqrt{R^2+1}}$.
*Hint:* Consider the line segment from $(0,0)$ to $(R,1)$. Calculate the distance from the closest integer points to the ray.

**Problem 22:**
Show that there is no line of sight through the orchard if $r > \frac{1}{R}$. You may want to use the following steps:

- Show that there is no line of sight if $r \geq 1$.
- Suppose $r < 1$ and $r > \frac{1}{R}$. Then, $R \geq 2$. Choose a potential line of sight passing through an arbitrary point $P$ on the circle. Thicken this line of sight equally on both sides into a rectangle of width $2r$ tangent to $P$ and $-P$. From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

**Problem 23: Challenge**
Prove that there exists a rational approximation of $\sqrt{3}$ within $10^{-3}$ with denominator at most 501. Come up with an upper bound for the smallest denominator of a $\epsilon$-close rational approximation of any irrational number $\alpha > 0$. Your bound can have some dependence on $\alpha$ and should get smaller as $\alpha$ gets larger.
*Hint:* Use the orchard.