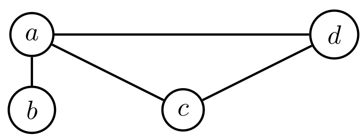
An Introduction to Graph Theory

Prepared by Mark on March 31, 2026
Based on a handout by Oleg Gleizer

Part 1: Graphs

Definition 1:
A *set* is an unordered collection of objects.
This means that the sets $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are identical.

Definition 2:
A *graph* $G = (N, E)$ consists of two sets: a set of *vertices* V, and a set of *edges* E.
Vertices are simply named "points," and edges are connections between pairs of vertices.
In the graph below, $V = \{a, b, c, d\}$ and $E = \{\ (a, b),\ (a, c),\ (a, d),\ (c, d)\ \}$.



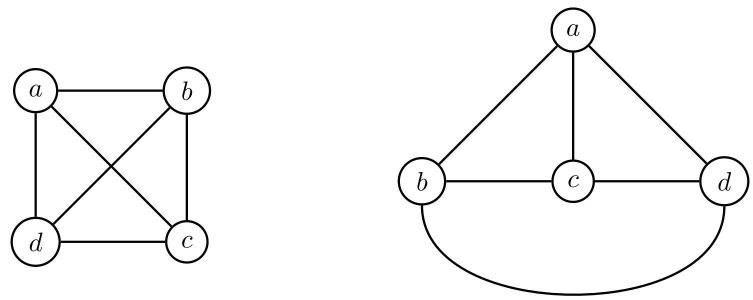
Vertices are also sometimes called *nodes*. You'll see both terms in this handout.

Problem 3:
Draw the graph defined by the following vertex and edge sets:
$V = \{A, B, C, D, E\}$
$E = \{\ (A, B),\ (A, C),\ (A, D),\ (A, E),\ (B, C),\ (C, D),\ (D, E)\ \}$

We can use graphs to solve many different kinds of problems.
Most situations that involve some kind of "relation" between elements can be represented by a graph.

Graphs are fully defined by their vertices and edges. The exact position of each vertex and edge doesn't matter—only which nodes are connected to each other. The same graph can be drawn in many different ways.

Problem 4:
Show that the graphs below are equivalent by comparing the sets of their vertices and edges.



Definition 5:
The degree $D(v)$ of a vertex v of a graph is the number of the edges of the graph connected to that vertex.

Theorem 6: Handshake Lemma
In any graph, the sum of the degrees of its vertices equals twice the number of the edges.

Problem 7:
Prove **??**.

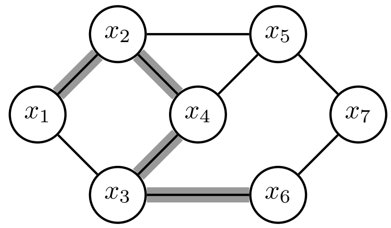
Problem 8:
Show that all graphs have an even number number of vertices with odd degree.

Problem 9:
One girl tells another, “There are 25 kids in my class. Isn’t it funny that each of them has 5 friends in the class?” “This cannot be true,” immediately replies the other girl. How did she know?

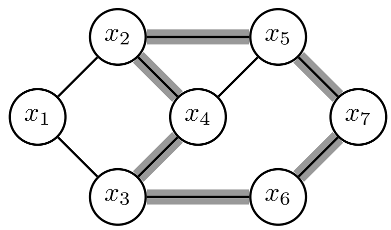
Problem 10:
Say G is a graph with nine vertices. Show that G has at least five vertices of degree six or at least six vertices of degree 5.

Part 2: Paths and cycles

A *path* in a graph is, intuitively, a sequence of edges: $(x_1, x_2, x_4, ...)$.
I've highlighted one possible path in the graph below.



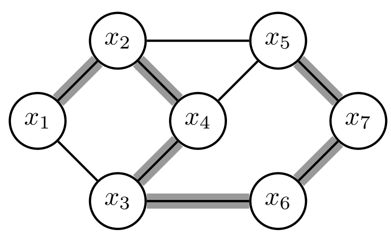
A *cycle* is a path that starts and ends on the same vertex:



A *Eulerian*[1] path is a path that traverses each edge exactly once.
A Eulerian cycle is a cycle that does the same.

Similarly, a *Hamiltonian* path is a path in a graph that visits each vertex exactly once, and a Hamiltonian cycle is a closed Hamiltonian path.

An example of a Hamiltonian path is below.

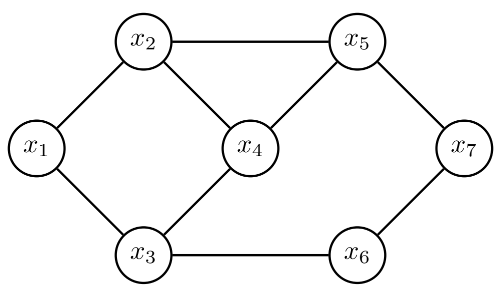


[1] Pronounced "oiler-ian". These terms are named after a Swiss mathematician, Leonhard Euler (1707-1783), who is usually considered the founder of graph theory.

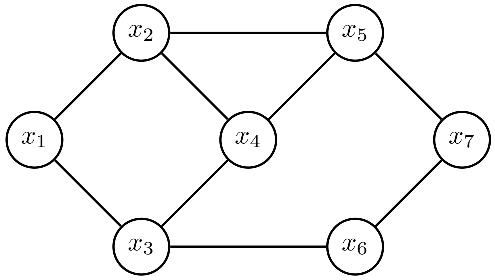
Definition 11:
We say a graph is *connected* if there is a path between every pair of vertices. A graph is called *disconnected* otherwise.

Problem 12:
Draw a disconnected graph with four vertices.
Then, draw a graph with four vertices, all of degree one.

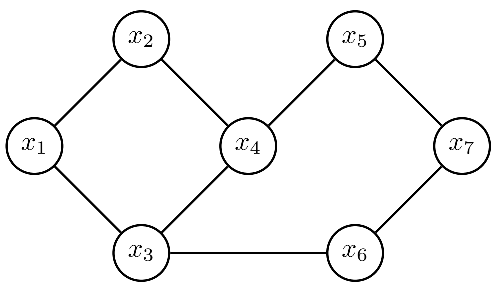
Problem 13:
Find a Hamiltonian cycle in the following graph.



Problem 14:
Is there an Eulerian path in the following graph?



Problem 15:
Is there an Eulerian path in the following graph?



Problem 16:
When does an Eulerian path exist?
Hint: Look at the degree of each node.

Part 3: Planar Graphs

TODO. Will feature planar graphs, euler's formula, utility problem, utility problem on a torus