

To Mock a Mockingbird

Prepared by Mark on February 10, 2025
Based on a book of the same name.

Part 1: Introduction

A certain enchanted forest is inhabited by talking birds. Each of these birds has a name, and will respond whenever it hears the name of another. Suppose you are exploring this forest and come across the bird A . You call the name of bird B . A hears you and responds with the name of some other bird, which we will designate AB .

Bird AB is, by definition, A 's response to B .

As you wander around this forest, you quickly discover two interesting facts:

A: A 's response to B mustn't be the same as B 's response to A .

B: Given three birds A , B , and C , $(AB)C$ and $A(BC)$ are not necessarily the same bird.

Bird $A(BC)$ is A 's response to bird BC , while $(AB)C$ is AB 's response to C .

Thus, ABC is ambiguous. Parenthesis are mandatory.

You also find that this forest has two laws:

A: *The Law of Composition:*

For any two birds A and B , there must be a bird C so that $Cx = A(Bx)$

B: *The Law of the Mockingbird:*

The forest must contain the Mockingbird M , which always satisfies $Mx = xx$.

In other words, the Mockingbird's response to any bird x is the same as x 's response to itself.

Definition 1:

We say a bird A is fond of a bird B if A responds to B with B .

In other words, A is fond of B if $AB = B$.

Definition 2:

We say a bird C composes A with B if for any bird x ,

$$Cx = A(Bx)$$

In other words, this means that C 's response to x is the same as A 's response to B 's response to x .

Note that C is exactly the kind of bird L_1 guarantees.

Part 2: To Mock a Mockingbird

Problem 3:

Mark tells you that any bird A is fond of at least one other bird.

Complete his proof.

```
let A           # Let A be any any bird.
let Cx = A(Mx)  # Define C as the composition of A and M

# The rest is up to you.
CC = ??
```

Things you will need:

Law: There exists a Mockingbird, $Mx := xx$

Def: A is fond of B if $AB = B$

Problem 4:

We say a bird A is *egocentric* if it is fond of itself.

Show that the laws of the forest guarantee that at least one bird is egocentric.

Things you will need:

Law: There exists a Mockingbird, $Mx := xx$

Def: A is fond of B if $AB = B$

Lem: Any bird is fond of at least one bird.

Definition 5:

We say a bird A is *agreeable* if for all birds B , there is at least one bird x on which A and B agree. In other words, A is agreeable if given any B , we can find a bird x satisfying $Ax = Bx$.

Problem 6:

Is the Mockingbird agreeable?

Problem 7:

Take two birds A and B . Let C be their composition.

Show that if C is agreeable, A is agreeable.

```
# Given information
```

```
let A, B
```

```
let Cx = A(Bx)
```

```
let D                # Arbitrary bird
```

```
let Ex = D(Bx)      # Define E as the composition of D and B
```

```
Cy = ??
```

Things you will need:

Def: A is agreeable if $Ax = Bx$ for all B with some x .

Law: For any A, B , there is C defined by $Cx = A(Bx)$

Problem 8:

Given three arbitrary birds A , B , and C , show that there exists a bird D satisfying $Dx = A(B(Cx))$

Definition 9:

We say two birds A and B are *compatible* if there are birds x and y so that $Ax = y$ and $By = x$. Note that x and y may be the same bird.

Problem 10:

Show that any two birds in this forest are compatible.

```
let A, B
let Cx = A(Bx)
```

Things you will need:

Law: Law of composition

Lem: Any bird is fond of at least one bird.

Problem 11:

Show that any bird that is fond of at least one bird is compatible with itself.

Part 3: The Curious Kestrel

Definition 12:

Recall that a bird is *egocentric* if it is fond of itself.

A bird is *hopelessly egocentric* if $Bx = B$ for all birds x .

Definition 13:

More generally, we say that a bird A is *fixated* on a bird B if $Ax = B$ for all x .

Convince yourself that a hopelessly egocentric bird is fixated on itself.

Problem 14:

Say A is fixated on B . Is A fond of B ?

Definition 15:

The *Kestrel* K is defined by the following relationship:

$$(Kx)y = x \quad \forall x, y$$

In other words, this means that for every bird x , the bird Kx is fixated on x .

Problem 16:

Show that an egocentric Kestrel is hopelessly egocentric.

Problem 17:

Assume the forest contains a Kestrel.

Given the Law of Composition and the Law of the Mockingbird, show that at least one bird is hopelessly egocentric.

Things you will need:

Def: K is defined by $(Kx)y = x$

Def: A is fond of B if $AB = B$

???: You'll need one more result from the previous section. Good luck!

Problem 18: Kestrel Left-Cancellation

In general, $Ax = Ay$ does not imply $x = y$. However, this is true if A is K .

Show that $Kx = Ky \implies x = y$.

This is a hint.

let x, y so that $Kx = Ky$

Problem 19:

Show that if K is fond of Kx , K is fond of x .

Problem 20:

An egocentric Kestrel must be extremely lonely. Why is this?