Pidgeonhole Problems

Prepared by Mark on January 24, 2025

Instructor's Handout

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Problem 1:
Is it possible to cover an equilateral triangle with two smaller equilateral triangles? Why or why not?

Solution

In order to completely cover an equilateral triangle, the two smaller triangles must cover all three vertices. Since the longest length of an equilateral triangle is one of its sides, a smaller triangle cannot cover more than one vertex. Therefore, we cannot completely cover the triangle with two smaller copies.
Bonus question: Can you cover a square with three smaller squares?

Problem 2:
You are given $n+1$ distinct integers.
Prove that at least two of them have a difference divisible by n.

Solution

$n \mid (a-b) \iff a \equiv b \pmod{n}$
Let $i_0 ... i_{n+1}$ be our set of integers. If we pick $i_0 ... i_{n+1}$ so that no two have a difference divisible by n, we must have $i_0 \not\equiv i_k \pmod{n}$ for all $1 \leq k \leq n+1$. There are n such i_k, and there are n equivalence classes mod n.
Therefore, either, $i_1 ... i_{n+1}$ must cover all equivalence classes mod n (implying that $i_0 \equiv i_k \pmod{n}$ for some k), or there exist two elements in $i_1 ... i_{n+1}$ that are equivalent mod n.
In either case, we can find a, b so that $a \equiv b \pmod{n}$, which implies that n divides $a - b$.

Problem 3:
You have an 8×8 chess board with two opposing corner squares cut off. You also have a set of dominoes, each of which is the size of two squares. Is it possible to completely cover the board with dominos, so that none overlap nor stick out?

Solution

A domino covers two adjacent squares. Adjacent squares have different colors.
If you remove two opposing corners of a chessboard, you remove two squares of the same color, and you're left with 32 of one and 30 of the other.
Since each domino must cover two colors, you cannot cover the modified board.

Problem 4:
The ocean covers more than a half of the Earth's surface. Prove that the ocean has at least one pair of antipodal points.

Solution

Let W be the set of wet points, and W^c the set of points antipodal to those in W. W and W^c each contain more than half of the points on the earth. The set of dry points, D, contains less than half of the points on the earth. Therefore, $W^c \not\subseteq D$.
Note: This solution isn't very convincing. However, it is unlikely that the students know enough to provide a fully rigorous proof.

Problem 5:
There are $n > 1$ people at a party. Prove that among them there are at least two people who have the same number of acquaintances at the gathering. (We assume that if A knows B, then B also knows A)

Solution

Assume that every attendee knows a different number of people. There is only one way this may happen: the most popular person knows $n-1$ people (that is, everyone but himself), the second-most popular knows $n-2$, etc. The least-popular person must then know 0 people.
This is impossible, since we know that someone must know $n-1$.
(Remember, "knowing" must be mutual.)

Problem 6:
Pick five points in $\mathbb{R}^2$ with integral coordinates. Show that two of these form a line segment that has an integral midpoint.

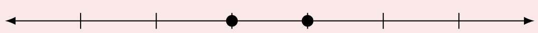
Solution

Let e, o represent even and odd integers.
There are four possible classes of points: (e, e), (o, o), (e, o), (o, e).
$\text{midpoint}(a, b) = (\frac{a_x+b_x}{2}, \frac{a_y+b_y}{2})$. If $a_x + b_x$ and $a_y + b_y$ are both even, the midpoint of points a and b will have integer coordinates.
Since we pick five points from four classes, at least two must come from the same class.
$e + e = e$ and $o + o = e$, so the midpoint between two points of the same class must have integral coordinates.

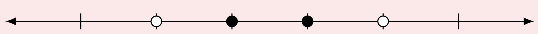
Problem 7:
Every point on a line is painted black or white. Show that there exist three points of the same color where one is the midpoint of the line segment formed by the other two.

Solution

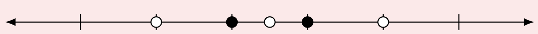
This is a proof by contradiction. We will try to construct a set of points where three points have such an arrangement.
We know that some two points on the line will have the same color:



This implies that the points one unit left and right of them must also be white—if they are not, they will form a line of equidistant black points.



Our original assumption also implies that the center point is white.
This, however, creates a line of equidistant white points:



It is thus impossible to create a set of points that does not have the property stated in the problem.

Problem 8:
Every point on a plane is painted black or white. Show that there exist two points in the plane that have the same color and are located exactly one foot away from each other.

Solution

Pick three points that form an equilateral triangle with side length 1.

Problem 9:
Let n be an integer not divisible by 2 and 5. Show that n has a multiple consisting entirely of ones.

Solution

Let $a_1 = 1, a_2 = 11, a_3 = 111$, and so on.

Consider the sequence $a_1, ..., a_{n+1}$.
By Problem 2, there exist a_i and a_j in this list so that $a_i - a_j \equiv 0 \pmod{n}$.

Since a_i and a_j are both made of ones, $a_i - a_j = 11...111 \times 10^j$.
n must be a factor of either 11...111 or 10^j.
Since n isn't divisible by 2 or 5, 10^j cannot be divisible by n, so 11...111 must be a factor of n.

Problem 10:
Prove that for any $n > 1$, there exists an integer made of only sevens and zeros that is divisible by n.

Solution

If n is not divisible by 2 or 5, the solution to this problem is the same as Problem 9:
just multiply the number of all ones by 7.

If n is divisible by 2 or 5, set p to the largest power of 2 or 5 in n.
Multiply the above number by 10^p to get a number that satisfies the conditions above.

Problem 11:
Choose $n + 1$ integers between 1 and $2n$. Show that at least two of these are co-prime.

Problem 12:
Choose $n + 1$ integers between 1 and $2n$. Show that you must select two numbers a and b such that a divides b.

Solution

Split the set $\{1, ..., 2n\}$ into classes defined by each integer's greatest odd divisor. There will be n classes since there are $\frac{k}{2}$ odd numbers between 1 and n. Because we pick $n + 1$ numbers, at least two will come from the same class—they will be divisible.
For example, if $n = 5$, our classes are
1: $\{1, 2, 4, 8\}$
3: $\{3, 6\}$
5: $\{5, 10\}$
7: $\{7\}$
9: $\{9\}$

Problem 13:
Show that it is always possible to choose a subset of the set of integers $\{a_1, a_2, ..., a_n\}$ so that the sum of the numbers in the subset is divisible by n.

Solution

Let $\{a'_1, a'_2, ..., a'_n\}$ be this set mod n.
If any a'_i is zero, we're done: $\{a'_i\}$ satisfies the problem.

If none are zero, consider the set $\{a'_1,\ a'_1 + a'_2,\ ...,\ a'_1 + a'_2 + ... + a'_n\}$.
If any element of this set is zero, we're done.

If zero is not in this set, we have n numbers with $n-1$ possible remainders. Therefore, at least two elements in this set must be equivalent mod n. If we subtract these two elements, we get a sum divisible by n.

Problem 14:
Show that there exists a positive integer divisible by 2013 that has 2014 as its last four digits.

Solution

Let n be this number.
First, note that $n - 2013$ has 0001 as its last four digits.

So, we see that $n - 2013 = 2013k \equiv 1 \pmod{1000}$.
Of course, $k = 2013^{-1} \pmod{1000}$, which exists because 2013 and 1000 are coprime.
And finally, we see that $n = 2013 \times (k + 1)$.

Problem 15:
Let n be an odd number. Let $a_1, a_2, ..., a_n$ be a permutation of the numbers $1, 2, ..., n$.
Show that $(a_1 - 1) \times (a_2 - 2) \times ... \times (a_n - n)$ is even.

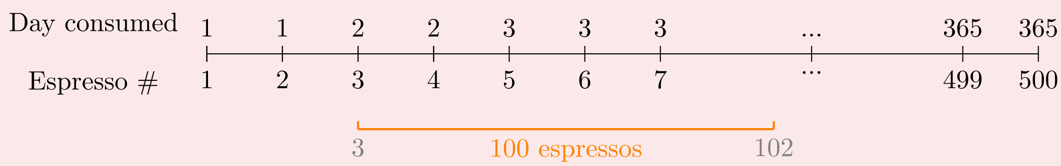
Solution

If n is odd, there will be m even and $m + 1$ odd numbers between 1 and n.
Therefore, if we match each a_n with an integer in $[1, ..., n]$, we will have to match at least one odd number with an odd number.
The difference of two odd numbers is even, so the product above will have at least one factor of two.

Problem 16:
A stressed-out student consumes at least one espresso every day of a particular year, drinking 500 overall. Show the student drinks exactly 100 espressos on some consecutive sequence of days.

Solution

Rearrange the problem. Don't think about days, think about espressos. Consider the following picture:



If there exists a sequence of days where the student drinks exactly 100 espressos, we must have at least one "block" (in orange, above) of 100 espressos that both begins and ends on a "clean break" between days.
There are 499 "breaks" between 500 espressos.
In a year, there are 364 clean breaks. This leaves $499 - 364 = 135$ "dirty" breaks.
We therefore have 135 places to start a block on a dirty break, and 135 places to end a block on a dirty break. This gives us a maximum of 270 dirty blocks.
However, there are 401 possible blocks, since we can start one at the $1^{\text{st}}, 2^{\text{nd}}, ..., 401^{\text{st}}$ espresso.
Out of 401 blocks, a maximum of 270 can be dirty. We are therefore guaranteed at least 131 clean blocks. This completes the problem—each clean block represents a set of consecutive, whole days during which exactly 100 espressos were consumed.

Problem 17:
Show that there are either three mutual acquaintances or four mutual strangers at a party with ten or more people.

Problem 18:
Given a table with a marked point, O, and with 2013 properly working watches put down on the table, prove that there exists a moment in time when the sum of the distances from O to the watches' centers is less than the sum of the distances from O to the tips of the watches' minute hands.