

To Mock a Mockingbird

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Based on a book of the same name.

Instructor's Handout

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Part 1: Introduction

A certain enchanted forest is inhabited by talking birds. Each of these birds has a name, and will respond whenever it hears the name of another. Suppose you are exploring this forest and come across the bird A . You call the name of bird B . A hears you and responds with the name of some other bird, which we will designate AB .

Bird AB is, by definition, A 's response to B .

As you wander around this forest, you quickly discover two interesting facts:

A: A 's response to B mustn't be the same as B 's response to A .

B: Given three birds A , B , and C , $(AB)C$ and $A(BC)$ are not necessarily the same bird.

Bird $A(BC)$ is A 's response to bird BC , while $(AB)C$ is AB 's response to C .

Thus, ABC is ambiguous. Parenthesis are mandatory.

You also find that this forest has two laws:

A: *The Law of Composition:*

For any two birds A and B , there must be a bird C so that $Cx = A(Bx)$

B: *The Law of the Mockingbird:*

The forest must contain the Mockingbird M , which always satisfies $Mx = xx$.

In other words, the Mockingbird's response to any bird x is the same as x 's response to itself.

Definition 1:

We say a bird A is fond of a bird B if A responds to B with B .

In other words, A is fond of B if $AB = B$.

Definition 2:

We say a bird C composes A with B if for any bird x ,

$$Cx = A(Bx)$$

In other words, this means that C 's response to x is the same as A 's response to B 's response to x . Note that C is exactly the kind of bird L_1 guarantees.

Part 2: To Mock a Mockingbird

Problem 3:

Mark tells you that any bird A is fond of at least one other bird.

Complete his proof.

```
let A          # Let A be any any bird.
let Cx = A(Mx) # Define C as the composition of A and M

# The rest is up to you.
CC = ??
```

Things you will need:

Law: There exists a Mockingbird, $Mx := xx$

Def: A is fond of B if $AB = B$

Solution

```
01 let A          # Let A be any any bird.
02 let Cx = A(Mx) # Define C as the composition of A and M
03 CC = A(MC)
04    = A(CC)    ■
```

Problem 4:

We say a bird A is *egocentric* if it is fond of itself.

Show that the laws of the forest guarantee that at least one bird is egocentric.

Things you will need:

Law: There exists a Mockingbird, $Mx := xx$

Def: A is fond of B if $AB = B$

Lem: Any bird is fond of at least one bird.

Solution

```
01 # We know M is fond of at least one bird.
02 let E so that ME = E
03
04 ME = E      # By definition of fondness
05 ME = EE     # By definition of M
06 ⇒ EE = E    ■
```

Definition 5:

We say a bird A is *agreeable* if for all birds B , there is at least one bird x on which A and B agree. In other words, A is agreeable if given any B , we can find a bird x satisfying $Ax = Bx$.

Problem 6:

Is the Mockingbird agreeable?

Solution

We know that $Mx = xx$.

From this definition, we see that M agrees with any x on x itself.

Problem 7:

Take two birds A and B . Let C be their composition.

Show that if C is agreeable, A is agreeable.

```
# Given information
let A, B
let Cx = A(Bx)

let D          # Arbitrary bird
let Ex = D(Bx) # Define E as the composition of D and B
Cy = ??
```

Things you will need:

Def: A is agreeable if $Ax = Bx$ for all B with some x .

Law: For any A, B , there is C defined by $Cx = A(Bx)$

Solution

```
01 # Given information
02 let A, B
03 let Cx = A(Bx)
04
05 let D          # Arbitrary bird
06 let Ex = D(Bx) # Define E as the composition of D and B
07 let y so that Cy = Ey # Such a y must exist because C is agreeable
08
09 A(By) = Ey
10      = D(By) ■
```

Problem 8:

Given three arbitrary birds A , B , and C , show that there exists a bird D satisfying $Dx = A(B(Cx))$

Solution

```

01 let A, B, C
02
03 # Invoke the Law of Composition:
04 let Qx = B(Cx)
05 let Dx = A(Qx)
06
07 Dx = A(Qx)
08     = A(B(Cx)) ■

```

Definition 9:

We say two birds A and B are *compatible* if there are birds x and y so that $Ax = y$ and $By = x$. Note that x and y may be the same bird.

Problem 10:

Show that any two birds in this forest are compatible.

```

let A, B
let Cx = A(Bx)

```

Things you will need:

Law: Law of composition
Lem: Any bird is fond of at least one bird.

Solution

```

01 let A, B
02
03 let Cx = A(Bx) # Composition
04 let y = Cy     # Let C be fond of y
05
06 Cy = y
07     = A(By)
08
09 let x = By    # Rename By to x
10 Ax = y ■

```

Problem 11:

Show that any bird that is fond of at least one bird is compatible with itself.

Solution

```

01 let A
02 let x so that Ax = x # A is fond of at least one other bird
03 Ax = x ■

```

Solution (continued)

That's it.

Part 3: The Curious Kestrel

Definition 12:

Recall that a bird is *egocentric* if it is fond of itself.

A bird is *hopelessly egocentric* if $Bx = B$ for all birds x .

Definition 13:

More generally, we say that a bird A is *fixated* on a bird B if $Ax = B$ for all x .

Convince yourself that a hopelessly egocentric bird is fixated on itself.

Problem 14:

Say A is fixated on B . Is A fond of B ?

Solution

Yes! See the following proof.

```
01 let A
02 let B so that Ax = B
03 ⇒ AB = B ■
```

Definition 15:

The *Kestrel* K is defined by the following relationship:

$$(Kx)y = x \quad \forall x, y$$

In other words, this means that for every bird x , the bird Kx is fixated on x .

Problem 16:

Show that an egocentric Kestrel is hopelessly egocentric.

Solution

```
01 KK = K
02 ⇒ (KK)y = K # By definition of the Kestrel
03 ⇒ Ky = K ■ # By 01
```

Problem 17:

Assume the forest contains a Kestrel.

Given the Law of Composition and the Law of the Mockingbird, show that at least one bird is hopelessly egocentric.

Things you will need:

Def: K is defined by $(Kx)y = x$

Def: A is fond of B if $AB = B$

???: You'll need one more result from the previous section. Good luck!

Solution

The final piece is a lemma we proved earlier:

Any bird is fond of at least one bird

```
01 let A so that KA = A    # Any bird is fond of at least one bird
02 (KA)y = y              # By definition of the kestrel
03 ⇒ Ay = A ■            # By 01
```

Problem 18: Kestrel Left-Cancellation

In general, $Ax = Ay$ does not imply $x = y$. However, this is true if A is K .

Show that $Kx = Ky \implies x = y$.

This is a hint.

let x, y so that Kx = Ky

Solution

```
01 let x, y so that Kx = Ky
02 let z
03
04 (Kx)z = (Ky)z    # By 01
05
06 # By the definition of K
07 (Kx)z = x
08 (Ky)z = y
09
10 ⇒ x = (Kx)z = (Ky)z = y ■
```

Problem 19:

Show that if K is fond of Kx , K is fond of x .

Solution

```

01 let x so that  $K(Kx) = Kx$ 
02  $(K(Kx))y = (Kx)y$ 
03           =  $Kx$  # By definition of  $K$ 
04  $x = Kx$  # By 03 and definition of  $K$ 

```

Problem 20:

An egocentric Kestrel must be extremely lonely. Why is this?

Solution

If a Kestrel is egocentric, it must be the only bird in the forest!

```

01 # Given
02  $Kx = K$  for some  $x$ 
03 # We have shown that an egocentric kestrel is hopelessly egocentric
04  $Kx = K$  for all  $x$ 
05
06 let  $x, y$ 
07  $Kx = K$ 
08  $Ky = K$ 
09  $Kx = Ky$ 
10  $x = y$  for all  $x, y$  # By Problem 18
11  $x = y = K$  ■ # By 10, and since  $K$  exists

```