

Definable Sets

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Instructor's Handout

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Part 1: Logical Algebra

Definition 1:

Logical operators operate on the values `{true, false}`, just like algebraic operators operate on numbers.

In this handout, we'll use the following operators:

- \neg : not
- \wedge : and
- \vee : or
- \rightarrow : implies
- $()$: parenthesis.

The function of these is defined by *truth tables*:

| and | | | or | | | implies | | | not | |
|-----|-----|--------------|-----|-----|------------|---------|-----|-------------------|-----|----------|
| A | B | $A \wedge B$ | A | B | $A \vee B$ | A | B | $A \rightarrow B$ | A | $\neg A$ |
| F | F | F | F | F | F | F | F | T | T | F |
| F | T | F | F | T | T | F | T | T | F | T |
| T | F | F | T | F | T | T | F | F | T | F |
| T | T | T | T | T | T | T | T | T | F | F |

$A \wedge B$ is **true** only if both A and B are **true**. $A \vee B$ is **true** if A or B (or both) are **true**.

$\neg A$ is the opposite of A , which is why it looks like a “negative” sign.

$A \rightarrow B$ is a bit harder to understand. Read aloud, this is “ A implies B .”

The only time \rightarrow produces **false** is when **true** \rightarrow **false**. This fact may seem counterintuitive, but will make more sense as we progress through this handout.

Hint: Think about it—if event α implies β , it is impossible for α to occur without β .

This is the only impossibility. All other variants are valid.

Problem 2:

Evaluate the following.

- $\neg T$
- $F \vee T$
- $T \wedge T$
- $(T \wedge F) \vee T$
- $(\neg(F \vee \neg T)) \rightarrow \neg T$
- $(F \rightarrow T) \rightarrow (\neg F \vee \neg T)$

Solution

F T T T F T

Note for Instructors

We can also think of $[x \geq 0] \rightarrow b$ as follows: if x isn't the kind of object we care about, we evaluate true and check the next one. If x *is* the kind of object we care about and b is false, we have a counterexample to $[x \geq 0] \rightarrow b$, and thus $\mathbf{T} \rightarrow \mathbf{F}$ must be false.

Say we have the sentence $\forall x (a \rightarrow b)$.

For example, take $\varphi = \forall x ([x \geq 0] \rightarrow [\exists y y^2 = x])$.

φ holds whenever any positive x has a square root.

If $(\mathbf{F} \rightarrow *)$ returned false, statements like the above would be hard to write.

If x is negative, φ doesn't care whether or not it has a root. In this case, $\mathbf{F} \rightarrow *$ must be true to avoid making whole \forall false.

Problem 3:

Evaluate the following.

- $A \rightarrow \mathbf{T}$ for any A
- $(\neg(A \rightarrow B)) \rightarrow A$ for any A, B
- $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ for any A, B

Note for Instructors

Note that the last formula is the contrapositive of $A \rightarrow B$.

Solution

All are true.

Problem 4:

Show that $\neg(A \rightarrow \neg B)$ is equivalent to $A \wedge B$.

That is, show that these expressions always evaluate to the same value given the same A and B .

Hint: Use a truth table

Problem 5:

Write an expression equivalent to $A \vee B$ using only \neg , \rightarrow , and $()$?

Solution

$((\neg A) \rightarrow B)$

Note that both \wedge and \vee can be defined using the other logical symbols.

The only logical symbols we *need* are \neg , \rightarrow , and $()$.

We include \wedge and \vee to simplify our expressions.

Part 2: Structures

Definition 6:

A *universe* is a set of meaningless objects. Here are a few examples:

- $\{a, b, \dots, z\}$
- $\{0, 1\}$
- \mathbb{Z}, \mathbb{R} , etc.

Definition 7:

A *structure* consists of a universe and a set of *symbols*.

A structure's symbols give meaning to the objects in its universe.

Symbols come in three types:

- *Constant symbols*, which let us specify specific elements of our universe.
Examples: $0, 1, \frac{1}{2}, \pi$
- *Function symbols*, which let us navigate between elements of our universe.
Examples: $+, \times, \sin x, \sqrt{x}$
Note that symbols we usually call “operators” are functions under this definition.
The only difference between $a + b$ and $+(a, b)$ is notation.
- *Relation symbols*, which let us compare elements of our universe.
Examples: $<, >, \leq, \geq$

The equality check $=$ is *not* a relation symbol. It is included in every structure by default.

By definition, $a = b$ is true if and only if a and b are the same element of our universe.

Example 8:

The first structure we'll look at is the following:

$$\left(\mathbb{Z} \mid \{0, 1, +, -, <\} \right)$$

This is a structure over the universe \mathbb{Z} that provides the following symbols:

- Constants: $\{0, 1\}$
- Functions: $\{+, -\}$
- Relations: $\{<\}$

If we look at our set of constant symbols, we see that the only integers we can directly refer to in this structure are 0 and 1. If we want any others, we must define them using the tools this structure offers.

To “define” an element of a set, we need to write a sentence that is only true for that element.

If we want to define 2 in the structure above, we could use the following sentence:

$$\text{“2 is the } x \text{ that satisfies } [1 + 1 = x].\text{”}$$

This is a valid definition because 2 is the *only* element of \mathbb{Z} for which $[1 + 1 = x]$ evaluates to **true**.

Problem 9:

Define -1 in $\left(\mathbb{Z} \mid \{0, 1, +, -, <\} \right)$.

Solution

The sentences “ x where $[x + 1 = 0]$ ” and “ x where $[0 - 1 = x]$ ” both work.

Let us formalize what we found in the previous two problems.

Definition 10: Formulas

A *formula* in a structure S is a well-formed string of constants, functions, relations, and logical operators.

You already know what a “well-formed string” is: $1 + 1$ is fine, $\sqrt{+}$ is nonsense.

For the sake of time, I will not provide a formal definition — it isn’t particularly interesting.

As a quick example, the formula $\psi := [\neg(1 = 1)]$ is always false,

and $\varphi(x) := [1 + 1 = x]$ evaluates to **true** only when x is 2.

Definition 11: Free Variables

A formula can contain one or more *free variables*. These are denoted $\varphi(a, b, \dots)$.

Formulas with free variables let us define “properties” that certain objects have.

For example, consider the two formulas from the previous definition, ψ and φ :

- $\psi := [\neg(1 = 1)]$
There are no free variables in this formula.
In any structure, ψ is always either **true** or **false**.
- $\varphi(x) := [1 + 1 = x]$
This formula has one free variable, labeled x .
The value of $\varphi(x)$ depends on the x we’re talking about:
 $\varphi(72)$ is false, and $\varphi(2)$ is true.

This “free variable” notation is very similar to the function notation we are used to:

The values of both $\varphi(x) := [x > 0]$ and $f(x) = x + 1$ depend on x .

Definition 12: Definable Elements

Let S be a structure over a universe U .

We say an element $x \in U$ is *definable in S* if we can write a formula $\varphi(x)$ that only x satisfies.

Problem 13:

Define 2 in the structure $(\mathbb{Z}^+ \mid \{4, \times\})$.

Hint: $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. Also, $2 \times 2 = 4$.

Solution

2 is the only element in \mathbb{Z}^+ that satisfies $\varphi(x) := [x \times x = 4]$.

Problem 14:

Try to define 2 in the structure $(\mathbb{Z} \mid \{4, \times\})$.

Why can't you do it?

Solution

We could try $\varphi(x) := [x \times x = 4]$, but this is satisfied by both 2 and -2 .

We have no way to distinguish between negative and positive numbers.

This problem is intentionally hand-wavy. We don't have the tools to write a proper proof.

Note for Instructors

Actually, it is. Bonus problem: how?

Do this after understanding quantifiers.

Problem 15:

Consider the structure $(\mathbb{R}_0^+ \mid \{1, 2, \div\})$

- Define 2^2
- Define 2^n for all positive integers n
- Define 2^{-n} for all positive integers n
- What other numbers can we define in this structure?

Hint: There is at least one more "class" of numbers we can define.

Solution

As far as I've seen, we can define any $2^{a/b}$ for $a, b \in \mathbb{Z}$.

For example, $\phi(x) := [2 = x \div (1 \div x)]$ defines $\sqrt{2}$.

Part 3: Quantifiers

Recall the logical symbols we introduced earlier: $()$, \wedge , \vee , \neg , \rightarrow
 We will now add two more: \forall (for all) and \exists (exists).

Definition 16:

\forall and \exists are *quantifiers*. They allow us to make statements about arbitrary symbols.
 Quantifiers are aptly named: they tell us *how many* symbols satisfy a certain sentence.

Let's look at \forall first. If $\varphi(x)$ is a formula,
 the formula $\forall x \varphi(x)$ is true only if φ is true for all x in our universe.

For example, take the formula $\forall x (0 < x)$.

In English, this means “For any x , x is bigger than zero,” or simply “Any x is positive.”

\exists is very similar: the formula $\exists x \varphi(x)$ is true if there is at least one x for which $\varphi(x)$ is true.
 For example, $\exists (0 < x)$ means “there is a positive number in our set.”

Problem 17:

Which of the following are true in \mathbb{Z} ? Which are true in \mathbb{R}_0^+ ?

\mathbb{R}_0^+ is the set of positive real numbers and zero.

- $\forall x (x \geq 0)$
- $\neg(\exists x (x = 0))$
- $\forall x [\exists y (y \times y = x)]$
- $\forall xy \exists z (x < z < y)$ This is a compact way to write $\forall x (\forall y (\exists z (x < z < y)))$
- $\neg \exists x (\forall y (x < y))$

Solution

- | | |
|--------------------------|------------------|
| • “all x are positive” | \mathbb{R}_0^+ |
| • “zero doesn’t exist” | neither |
| • “square roots exist” | \mathbb{R}_0^+ |
| • “this set is dense” | \mathbb{R}_0^+ |
| • “there is no minimum” | \mathbb{Z} |

Problem 18:

Does the order of \forall and \exists in a formula matter?

What's the difference between $\exists x \forall y (x \leq y)$ and $\forall y \exists x (x \leq y)$?

Hint: Consider \mathbb{R}^+ , the set of positive reals. Zero is not positive.

Which of the above formulas is true in \mathbb{R}^+ and which is false?

Solution

If $\exists x$ is inside $\forall y$, x depends on y . We may pick a different value of x for every y .

If $\exists x$ is outside, x is fixed *before* we check all y .

Problem 19:

Define 0 in $(\mathbb{Z} \mid \{\times\})$

Solution

$$\varphi(x) := [\forall y \ x \times y = x]$$

Problem 20:

Define 1 in $(\mathbb{Z} \mid \{\times\})$

Solution

$$\varphi(x) := [\forall y \ x \times y = y]$$

Problem 21:

Define -1 in $(\mathbb{Z} \mid \{0, <\})$

Solution

$$\varphi(x) := [(x < 0) \wedge \neg \exists y (x < y < 0)]$$

Problem 22:

Let $\varphi(x)$ be a formula.

Write a formula equivalent to $\forall x \varphi(x)$ using only logical symbols and \exists .

Solution

$\forall x \varphi(x)$ is true if and only if $\neg \exists x \neg \varphi(x)$ is true.

Part 4: Definable Sets

Armed with $()$, \wedge , \vee , \neg , \rightarrow , \forall , and \exists , we have the tools to define sets.

Definition 23: Set-Builder Notation

Say we have a sentence $\varphi(x)$.

The set of all elements that satisfy that sentence may be written as follows:

$$\{x \mid \varphi(x)\}$$

This is read “The set of x where φ is true” or “The set of x that satisfy φ .”

For example, take the formula $\varphi(x) = \exists y (y + y = x)$.

The set of all even integers can then be written as

$$\{x \mid \exists y (y + y = x)\}$$

Definition 24: Definable Sets

Let S be a structure with a universe U .

We say a subset M of U is *definable* if we can write a formula that is true for some x if and only if M contains x .

For example, consider the structure $(\mathbb{Z} \mid \{+\})$.

Only even numbers satisfy the formula $\varphi(x) := [\exists y (y + y = x)]$, so we can define “the set of even numbers” as $\{x \mid \exists y (y + y = x)\}$.

Remember—we can only use symbols that are available in our structure!

Problem 25:

The empty set is definable in any structure. How?

Solution

Always: $\{x \mid \neg(x = x)\}$

Problem 26:

Define $\{0, 1\}$ in $(\mathbb{Z}_0^+ \mid \{<\})$ *Hint:* Define 0 and 1 as elements first, and remember that we can use logical symbols.

Solution

$$\varphi_0(x) := [\neg \exists y y < x]$$

$$\varphi_1(x) := [(0 < x) \wedge \neg \exists y (x < y < 0)]$$

Our final solution is $\{x \mid \varphi_0(x) \vee \varphi_1(x)\}$.

A finite set of definable elements is always definable.

An infinite set of definable elements might not be definable.

Problem 27:

Define the set of prime numbers in $(\mathbb{Z} \mid \{\times, \div, <\})$.

Hint: A prime number is an integer that is positive and is only divisible by 1 and itself.

Solution

$$\psi(x) := [\exists y (0 < y < x)] \quad \text{“}x \text{ is positive and isn’t 0 or 1”}$$

$$\varphi(x) := [(x < 0) \wedge \neg \exists ab (\psi(a) \wedge \psi(b) \wedge a \times b = x)]$$

Problem 28:

Define \mathbb{R}_0^+ in $(\mathbb{R} \mid \{\times\})$

Solution

$$\varphi(x) := [\exists y \ y \times y = x]$$

Problem 29:

Let \triangle be a relational symbol. $a \triangle b$ is only true if a divides b .

Define the set of prime numbers in $(\mathbb{Z}^+ \mid \{\triangle\})$

Solution

$$\varphi(x) := [\neg \exists abc \ ((a \triangle x) \wedge (b \triangle x) \wedge (c \triangle x) \wedge \neg(a = b) \wedge \neg(a = c) \wedge \neg(b = c))]$$

Theorem 30: Lagrange's Four Square Theorem

Every natural number may be written as a sum of four integer squares.

Problem 31:

Define \mathbb{Z}_0^+ in $(\mathbb{Z} \mid \{\times, +\})$

Solution

$$\varphi(x) := [\exists abcd \ (a^2 + b^2 + c^2 + d^2 = x)], \text{ where } a^2 := a \times a.$$

Problem 32:

Define $<$ in $(\mathbb{Z} \mid \{\times, +\})$

Hint: We can't formally define a relation yet. Don't worry about that for now.

You can rephrase this question as “given $x, y \in \mathbb{Z}$, write a formula $\varphi(x, y)$ that is only true if $x < y$ ”

Solution

Let $\psi(x)$ be the formula from the previous problem.

$$\varphi(x, y) := [\neg(x = y) \wedge \exists d \ (\psi(d) \wedge (x + d = y))]$$

Problem 33:

Consider the structure $S = (\mathbb{R} \mid \{0, \diamond\})$

The relation $a \diamond b$ holds if $|a - b| = 1$

Part 1:

Define $\{-1, 1\}$ in S .

Solution

$$\varphi(x) := [0 \diamond x]$$

Part 2:

Define $\{-2, 2\}$ in S .

Solution

$$\varphi(x) := [\forall a (0 \diamond x \rightarrow a \diamond x) \wedge \neg(x = 0)]$$

Problem 34:

Let \mathcal{P} be the set of all subsets of \mathbb{Z}_0^+ . This is called the *power set* of \mathbb{Z}_0^+ .

Let S be the structure $(\mathcal{P} \mid \{\subseteq\})$

Part 1:

Show that the empty set is definable in S .

Hint: Defining $\{\}$ with $\{x \mid \neg x = x\}$ is **not** what we need here.

We need $\emptyset \in \mathcal{P}$, the “empty set” element in the power set of \mathbb{Z}_0^+ .

Solution

$$\varphi(x) := [\forall y x \subseteq y]$$

Note that we can use the same property to define 0 in $(\mathbb{Z} \mid \{\leq\})$

Part 2:

Let $x \vartriangleleft y$ be a relation on \mathcal{P} . $x \vartriangleleft y$ holds if $x \cap y \neq \{\}$.

Show that \vartriangleleft is definable in S .

Solution

Let $\psi(x)$ be the formula from the previous problem.

$$\varphi(x, y) := [\exists x (a \subseteq x) \wedge (a \subseteq y) \wedge \neg\psi(a)]$$

Part 3:

Let f be the function on \mathcal{P} defined by $f(x) = \mathbb{Z}_0^+ - x$. This is called the *complement* of x .

Show that f is definable in S .

Hint: You can define a function by writing a formula $\varphi(x, y)$ that is only true when $y = f(x)$.

Part 5: Equivalence

Notation:

Let S be a structure and φ a formula.

If φ is true in S , we write $S \models \varphi$.

This is read “ S satisfies φ ”

Definition 35:

Let S and T be structures.

We say S and T are *equivalent* (and write $S \equiv T$) if for any formula φ , $S \models \varphi \iff T \models \varphi$.

If S and T are not equivalent, we write $S \not\equiv T$.

Problem 36:

Show that $(\mathbb{Z} \mid \{+, 0\}) \not\equiv (\mathbb{R} \mid \{+, 0\})$

Problem 37:

Show that $(\mathbb{Z} \mid \{+, 0\}) \not\equiv (\mathbb{N} \mid \{+, 0\})$

Problem 38:

Show that $(\mathbb{R} \mid \{+, 0\}) \not\equiv (\mathbb{N} \mid \{+, 0\})$

Problem 39:

Show that $(\mathbb{R} \mid \{+, 0\}) \not\equiv (\mathbb{Z}^2 \mid \{+, 0\})$

Problem 40:

Show that $(\mathbb{Z} \mid \{+, 0\}) \not\equiv (\mathbb{Z}^2 \mid \{+, 0\})$

Solution

All of the above are easy, but the last one can take a while.

The trick is to notice that \mathbb{Z} has two equivalence classes mod 2, while \mathbb{Z}^2 has four.