

# Geometric Optimization

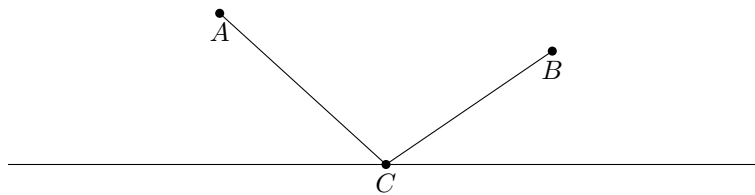
Prepared by Mark on January 24, 2025  
Based on a handout by Nakul & Andreas

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## Part 1: Optimization

**Problem 1:**

Let  $A$  and  $B$  be two points on the same side of a given line  $\ell$ .  
Find a point  $C$  on  $\ell$  so that  $|AC| + |BC|$  is minimized.



**Definition 1:**

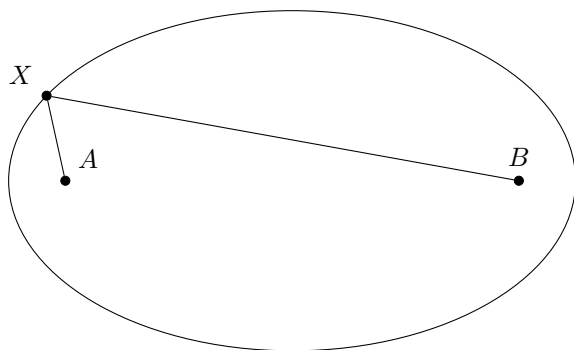
An *ellipse* with foci  $A$ ,  $B$  and radius  $r$  is the set of all points  $C$  where  $|AB| + |BC| = r$ .

**Problem 2:**

Consider a reflective ellipse with foci  $A$  and  $B$ .

Find all points  $X$  on the ellipse where  $A$  can aim a laser at so that the beam reaches  $B$ .

*Hint:* use ??



**Problem 3:**

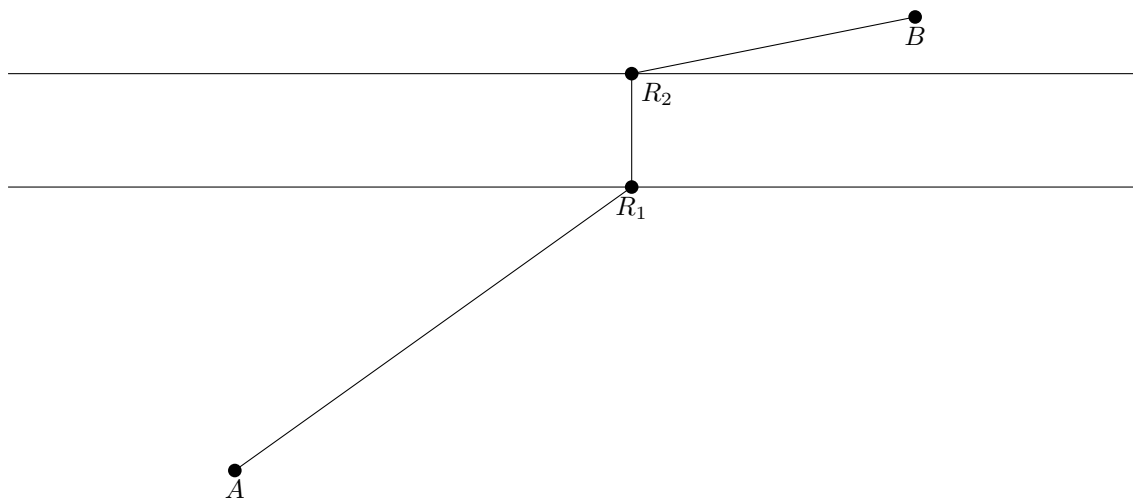
Let  $C$  be a point in the interior of a given angle. Find points  $A$  and  $B$  on the sides of the angle such that the perimeter of the triangle  $ABC$  is a minimum.

**Problem 4:**

In a convex quadrilateral  $ABCD$ , find the point  $T$  for which the sum of the distances to the vertices is minimal.

**Problem 5:**

A road needs to be constructed from town A to town B, crossing a river, over which a perpendicular bridge is to be constructed. Where should the bridge be placed to minimize  $|AR_1| + |R_1R_2| + |R_2B|$ ?

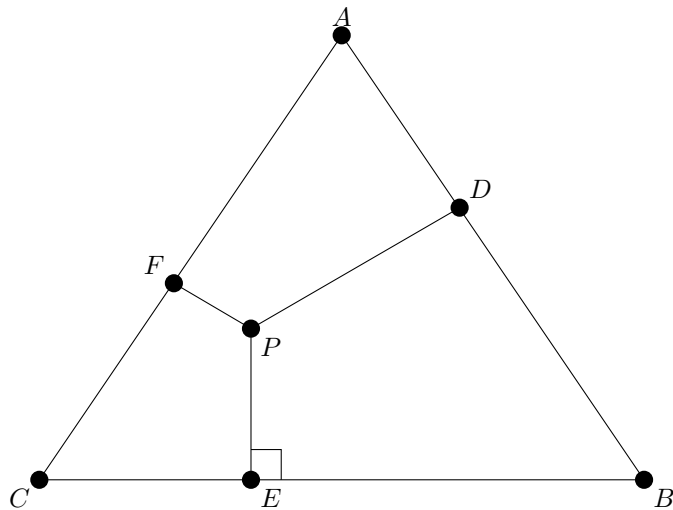


**Problem 6:**

Consider an equilateral triangle with vertices labeled  $A$ ,  $B$ , and  $C$ .

Let  $P$  be a point inside this triangle. Place  $D$ ,  $E$ , and  $F$  so that  $PD$ ,  $PE$ , and  $PF$  are the perpendiculars from  $P$  to the sides of the triangle.

Find all points  $P$  where  $|PD| + |PE| + |PF|$  is minimized.



**Problem 7:**

With the same setup as ??, find all points  $P$  where  $|PA| + |PB| + |PC|$  is minimized.

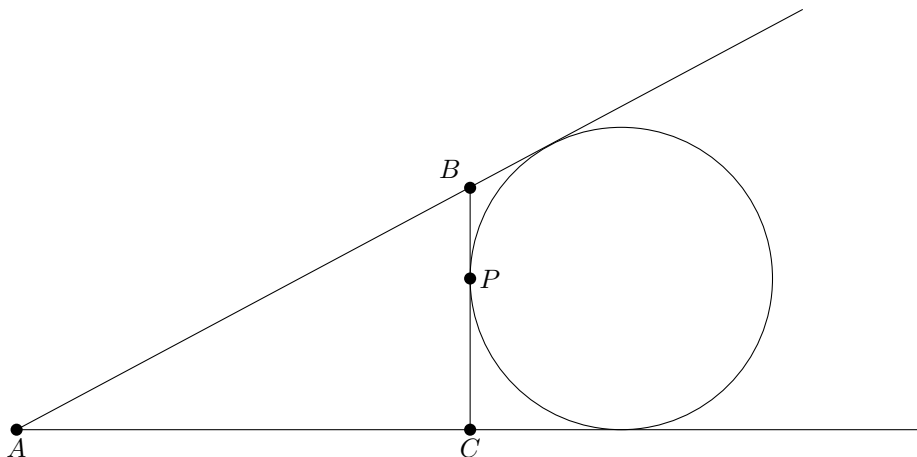
**Problem 8:**

Solve ?? for a triangle that isn't equilateral.

**Problem 9:**

Draw a circle, then draw two distinct tangents  $\ell_1$  and  $\ell_2$  that intersect at point  $A$ .

Let  $P$  be a point on the circle between the tangents, and  $BC$  be the tangent at that point. Describe how  $P$  should be selected in order to minimize the perimeter of triangle  $ABC$ .

**Problem 10:**

Now, assume that  $\ell_1$  and  $\ell_2$  intersect at  $A$ , and pick a point  $P$  between them.

Find  $BC$  through  $P$  so that the perimeter of  $ABC$  is minimized.

## Part 2: Bonus Problems

**Problem 11:**

Given a cube  $A_1B_1C_1D_1A_2B_2C_2D_2$  with side length  $l$ , find the angle and distance between lines  $A_1B_2$  and  $A_2C_1$ .

**Problem 12:**

Consider a cube  $A_1B_1C_1D_1A_2B_2C_2D_2$ , and let  $K$ ,  $L$ , and  $M$  be midpoints of the edges  $A_2D_2$ ,  $A_1B_1$ , and  $C_1C_2$ . Show that the triangle formed by  $KLM$  is equilateral, and that its center is the center of the cube.



**Problem 13:**

Consider all  $n$ -gons with a certain perimeter. Show that the  $n$ -gon with maximal area has equal sides

**Problem 14:**

Consider all  $n$ -gons with a certain perimeter. Show that the  $n$ -gon with maximal area has equal angles