Generating Functions

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Based on a handout by Aaron Anderson

Instructor's Handout

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Part 1: Introduction

Definition 1:
Say we have a sequence $a_0, a_1, a_2, ...$
The *generating function* of this sequence is defined as follows:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

Under some circumstances, this sum does not converge, and thus $A(x)$ is undefined.
However, we can still manipulate this infinite sum to get useful results even if $A(x)$ diverges.

Problem 2:
Let $A(x)$ be the generating function of the sequence a_n,
and let $B(x)$ be the generating function of the sequence b_n.
Find the sequences that correspond to the following generating functions:

- $cA(x)$
- $xA(x)$
- $A(x) + B(x)$
- $A(x)B(x)$

Solution

- $cA(x)$ corresponds to ca_n
- $xA(x)$ corresponds to $0, a_0, a_1, ...$
- $A(x) + B(x)$ corresponds to $a_n + b_n$
- $A(x)B(x)$ is $a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + ...$
  Which corresponds to $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Problem 3:
Assuming $|x| < 1$, show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

Hint: use some clever algebra. What is $x \times (1 + x + x^2 + ...)$?

Solution

Let $S = 1 + x + x^2 + ...$
Then, $xS = x + x^2 + x^3 + ...$

So, $xS = S - 1$
and $1 = S - xS = S(1 - x)$
and $S = \frac{1}{1-x}$.

Problem 4:
Let $A(x)$ be the generating function of the sequence a_n.
Find the sequence that corresponds to the generating function $\frac{A(x)}{1-x}$

Solution

$$\begin{aligned} \frac{A(x)}{1-x} &= A(x)(1 + x + x^2 + ...) \\ &= (a_0 + a_1 x + a_2 x^2 + ...)(1 + x + x^2 + ...) \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + ... \end{aligned}$$

Which corresponds to the sequence $c_n = \sum_{k=0}^{n} a_k$

Problem 5:
Find short expressions for the generating functions for the following sequences:

- $1, 0, 1, 0, ...$
- $1, 2, 4, 8, 16, ...$
- $1, 2, 3, 4, 5, ...$

Solution

- $1, 0, 1, 0, ...$ corresponds to $1 + x^2 + x^4 + ...$
  By Problem 3, this is $\frac{1}{1-x^2}$.
- $1, 2, 4, 8, 16, ...$ corresponds to $1 + 2x + (2x)^2 + ...$
  By Problem 3, this is $\frac{1}{1-2x}$.
- $1, 2, 3, 4, 5, ...$ corresponds to $1 + 2x + 3x^2 + 4x^3 + ...$
  This is equal to $(1 + x + x^2 + ...)^2$, and thus is $\left(\frac{1}{1-x}\right)^2$

Part 2: Fibonacci

Definition 6:
The *Fibonacci numbers* are defined by the following recurrence relation:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$

Problem 7:
Let $F(x)$ be the generating function that corresponds to the Fibonacci numbers.
Find the generating function of $0, f_0, f_1, ...$ in terms of $F(x)$.
Call this $G(x)$.

Solution

$$G(x) = xF(x)$$

Problem 8:
Find the generating function of $0, 0, f_0, f_1, ...$ in terms of $F(x)$. Call this $H(x)$.

Solution

$$H(x) = x^2 F(x)$$

Problem 9:
Calculate $F(x) - G(x) - H(x)$ using the recurrence relation that we used to define the Fibonacci numbers.

Solution

$$\begin{aligned} F(x) - G(x) - H(x) &= f_0 + (f_1 - f_0)x + (f_2 - f_1 - f_0)x^2 + (f_3 - f_2 - f_1)x^3 + ... \\ &= f_0 + (f_1 - f_0)x \\ &= x \end{aligned}$$

Problem 10:
Using the problems on the previous page, find $F(x)$ in terms of x.

Solution

$$\begin{aligned} x &= F(x) - G(x) - H(x) \\ &= F(x) - xF(x) - x^2F(x) \\ &= F(x)(1 - x - x^2) \end{aligned}$$

So,

$$F(x) = \frac{x}{1 - x - x^2}$$

Definition 11:
A *rational function* f is a function that can be written as a quotient of polynomials.
That is, $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials.

Problem 12:
Solve the equation from Problem 10 for $F(x)$, expressing it as a rational function.

Solution

$$\begin{aligned} F(x) &= \frac{-x}{x^2 + x - 1} = \frac{-x}{(x-a)(x-b)} \\ &= \frac{1-\sqrt{5}}{2\sqrt{5}} \frac{1}{x-a} + \frac{-1-\sqrt{5}}{2\sqrt{5}} \frac{1}{x-b} \end{aligned}$$

where

$$a = \frac{-1+\sqrt{5}}{2}; \quad b = \frac{-1-\sqrt{5}}{2}$$

Definition 13:
Partial fraction decomposition is an algebraic technique that works as follows:
If $p(x)$ is a polynomial of degree 1 and a and b are constants, we can rewrite the rational function $\frac{p(x)}{(x-a)(x-b)}$ as follows:

$$\frac{p(x)}{(x-a)(x-b)} = \frac{c}{x-a} + \frac{d}{x-b}$$

where c and d are constants.

Problem 14:
Now that we have a rational function for $F(x)$,
find a closed-form expression for its coefficients using partial fraction decomposition.

Solution

$$\begin{aligned} F(x) &= \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{-1}{a}\right)\left(\frac{1}{1-\frac{x}{a}}\right) + \left(\frac{-1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{-1}{b}\right)\left(\frac{1}{1-\frac{x}{b}}\right) \\ &= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1}{1-\frac{x}{a}}\right) + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1}{1-\frac{x}{b}}\right) \\ &= \frac{1}{\sqrt{5}}\left(1 + \frac{x}{a} + \left(\frac{x}{a}\right)^2 + \ldots\right) - \frac{1}{\sqrt{5}}\left(1 + \frac{x}{b} + \left(\frac{x}{b}\right)^2 + \ldots\right) \end{aligned}$$

Problem 15:
Using problems from the introduction and Problem 14, find an expression for the coefficients of $F(x)$ (and thus, for the Fibonacci numbers).

Solution

$$\begin{aligned} f_0 &= \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0 \\ f_1 &= \frac{1}{a\sqrt{5}} - \frac{1}{b\sqrt{5}} = 1 \\ f_n &= \frac{1}{\sqrt{5}}\left(\frac{1}{a^n} - \frac{1}{b^n}\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \end{aligned}$$

Problem 16: Bonus

Repeat the method of recurrence, generating function, partial fraction decomposition, and geometric series to find a closed form for the following sequence:

$$a_0 = 1; \quad a_{n+1} = 2a_n + n$$

Hint: When doing partial fraction decomposition with a denominator of the form $(x-a)^2(x-b)$, you may need to express your expression as a sum of three fractions: $\frac{c}{(x-a)^2} + \frac{d}{x-a} + \frac{e}{x-b}$.''

Part 3: Dice

Definition 17:
A *die* is a device that randomly selects a positive integer from a finite list of options. For example, the standard 6-sided die selects a value from $[1, 2, 3, 4, 5, 6]$. We may have many sides with the same value, as in $[1, 1, 2, 3]$.
To describe a die with a generating function, let a_k be the number of times k appears as a side of the die and consider $a_0 + a_1 x + x_2 x^2 + ...$
A die has a finite number of sides, so this will be a regular polynomial.

Problem 18:
What is the generating function of the standard 6-sided die?

Solution

$x + x^2 + x^3 + x^4 + x^5 + x^6$

Problem 19:
What is the generating function of the die with sides $[1, 2, 3, 5]$?

Solution

$2x + x^2 + x^3 + x^5$

Problem 20:
Let $A(x)$ and $B(x)$ be the generating functions of two dice.
What is the significance of $A(1)$?

Solution

$A(1) =$ the number of sides on the die

Problem 21:
Using formulas we found earlier, show that the k^{th} coefficient of $A(x)B(x)$ is the number of ways to roll k as the sum of the two dice.

Solution

The k^{th} coefficient of $A(x)B(x)$ is...

$$\begin{aligned} &a_0 b_k + a_1 b_{k+1} + ... + a_k b_0 \\ &\quad = \text{count}(A = 0; B = k) + ... + \text{count}(A = k; B = 0) \\ &\quad = \text{number of ways} A + B = k \end{aligned}$$

Problem 22:
Find a generating function for the sequence $c_0, c_1, ...$, where c_k is the probability that the sum of the two dice is k.

Solution

$$c_k = \frac{\text{number of ways sum} = k}{\text{number of total outcomes}} = \frac{\text{number of ways sum} = k}{A(1)B(1)}$$

So,

$$c_0 + c_1 x + c_2 x^2 = \frac{A(x)B(x)}{A(1)B(1)}$$

Problem 23:
Using generating functions, find two six-sided dice whose sum has the same distribution as the sum of two standard six-sided dice.
That is, for any integer k, the number if ways that the sum of the two nonstandard dice rolls as k is equal to the number of ways the sum of two standard dice rolls as k. *Hint:* factor polynomials.

Solution

We need a different factorization of

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = A(x)B(x)$$

We can use

$$(x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8)$$

Part 4: Coins

Consider the following problem:
"How many different ways can you make change for \$0.50
using pennies, nickels, dimes, quarters and half-dollars?"

Most ways of solving this involve awkward brute-force approaches that don't reveal anything interesting about the problem: how can we change our answer if we want to make change for \$0.51, or \$1.05, or some other quantity?

We can use generating functions to solve this problem in a general way.

Definition 24:
Let $p_0, p_1, p_2, ...$ be such that p_k is the number of ways to make change for k cents with only pennies.
Similarly, let...

- n_k be the number of ways to make change for k cents with only nickels;
- d_k be the number of ways using only dimes;
- q_k be the number of ways using only quarters;
- and h_k be the number of ways using only half-dollars.

Problem 25:
Let $p(x)$ be the generating function that corresponds to p_n.
Express $p(x)$ as a rational function.

Problem 26:
Modify Problem 25 to find expressions for $n(x)$, $d(x)$, $q(x)$, and $h(x)$.

Definition 27:
Now, let $N(x)$ be the generating function for the sequence $n_0, n_1, ...$, where n_k is the number of ways to make change for k cents using pennies and nickels.
Similarly, let...

- let $D(x)$ be the generating function for the sequence using pennies, nickels, and dimes;
- let $Q(x)$ use pennies, nickels, dimes, and quarters;
- and let $H(x)$ use all coins.

Problem 28:
Express $N(x)$ as a rational function.

Problem 29:
Using the previous problem, write $D(x)$, then $Q(x)$, then $H(x)$ as rational functions.

Problem 30:
Using these generating functions, find recurrence relations for the sequences N_k, D_k, Q_k, and H_k.
Hint: Your recurrence relation for N_k should refer to the previous values of itself and some values of p_k. Your recurrence for D_k should refer to itself and N_k; the one for Q_k should refer to itself D_k; and the one for H_k should refer to itself and Q_k.

Problem 31:
Using these recurrence relations, fill following table and solve the original problem.

n	0	5	10	15	20	25	30	35	40	45	50
p_k											
N_k											
D_k											
Q_k											
H_k											

Part 5: Extra Problems

Problem 32: USAMO 1996 Problem 6
Determine (with proof) whether there is a subset X of the nonnegative integers with the following property: for any nonnegative integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$. (The original USAMO question asked about all integers, not just nonnegative - this is harder, but still approachable with generating functions.)

Problem 33: IMO Shortlist 1998
Let $a_0, a_1, ...$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine $a_1 998$.

Problem 34: USAMO 1986 Problem 5
By a partition π of an integer $n \geq 1$, we mean here a representation of n as a sum of one or more positive integers where the summands must be put in nondecreasing order. (e.g., if $n = 4$, then the partitions π are $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3, 2 + 2$, and 4).
For any partition π, define $A(\pi)$ to be the number of ones which appear in π, and define $B(\pi)$ to be the number of distinct integers which appear in π (e.g, if $n = 13$ and π is the partition $1 + 1 + 2 + 2 + 2 + 5$, then $A(\pi) = 2$ and $B(\pi) = 3$).
Show that for any fixed n, the sum of $A(\pi)$ over all partitions of π of n is equal to the sum of $B(\pi)$ over all partitions of π of n.

Problem 35: USAMO 2017 Problem 2
Let $m_1, m_2, ..., m_n$ be a collection of n distinct positive integers. For any sequence of integers $A = (a_1, ..., a_n)$ and any permutation $w = w_1, ..., w_n$ of $m_1, ..., m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $ai \geq wi > wj$
- $wj > ai \geq wi$
- $wi > wj > ai$

Show that for any two sequences of integers $A = (a_1, ..., a_n)$ and $B = (b_1, ..., b_n)$ and for any positive integer k, the number of permutations of $m_1, ..., m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, ..., m_n$ having exactly k B-inversions. (The original USAMO problem allowed the numbers $m_1, ..., m_n$ to not necessarily be distinct.)