# Generating Functions

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Based on a handout by Aaron Anderson

## Part 1: Introduction

**Definition 1:**
Say we have a sequence $a_0, a_1, a_2, ....$
The *generating function* of this sequence is defined as follows:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

Under some circumstances, this sum does not converge, and thus $A(x)$ is undefined.
However, we can still manipulate this infinite sum to get useful results even if $A(x)$ diverges.

**Problem 2:**
Let $A(x)$ be the generating function of the sequence $a_n$,
and let $B(x)$ be the generating function of the sequence $b_n$.
Find the sequences that correspond to the following generating functions:

- $cA(x)$
- $xA(x)$
- $A(x) + B(x)$
- $A(x)B(x)$

**Problem 3:**
Assuming $|x| < 1$, show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

*Hint:* use some clever algebra. What is $x \times (1 + x + x^2 + ...)$?

**Problem 4:**
Let $A(x)$ be the generating function of the sequence $a_n$.
Find the sequence that corresponds to the generating function $\frac{A(x)}{1-x}$

**Problem 5:**
Find short expressions for the generating functions for the following sequences:

- $1, 0, 1, 0, ...$
- $1, 2, 4, 8, 16, ...$
- $1, 2, 3, 4, 5, ...$

# Part 2: Fibonacci

**Definition 6:**
The *Fibonacci numbers* are defined by the following recurrence relation:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$

**Problem 7:**
Let $F(x)$ be the generating function that corresponds to the Fibonacci numbers.
Find the generating function of $0, f_0, f_1, ...$ in terms of $F(x)$.
Call this $G(x)$.

**Problem 8:**
Find the generating function of $0, 0, f_0, f_1, ...$ in terms of $F(x)$. Call this $H(x)$.

**Problem 9:**
Calculate $F(x) - G(x) - H(x)$ using the recurrence relation that we used to define the Fibonacci numbers.

**Problem 10:**
Using the problems on the previous page, find $F(x)$ in terms of $x$.

**Definition 11:**
A *rational function* $f$ is a function that can be written as a quotient of polynomials. That is, $f(x) = \frac{p(x)}{q(x)}$ where $p$ and $q$ are polynomials.

**Problem 12:**
Solve the equation from **??** for $F(x)$, expressing it as a rational function.

**Definition 13:**
*Partial fraction decomposition* is an algebraic technique that works as follows:
If $p(x)$ is a polynomial of degree 1 and $a$ and $b$ are constants, we can rewrite the rational function $\frac{p(x)}{(x-a)(x-b)}$ as follows:

$$\frac{p(x)}{(x-a)(x-b)} = \frac{c}{x-a} + \frac{d}{x-b}$$

where $c$ and $d$ are constants.

**Problem 14:**
Now that we have a rational function for $F(x)$,
find a closed-form expression for its coefficients using partial fraction decomposition.

**Problem 15:**
Using problems from the introduction and **??**, find an expression for the coefficients of $F(x)$ (and thus, for the Fibonacci numbers).

**Problem 16: Bonus**

Repeat the method of recurrence, generating function, partial fraction decomposition, and geometric series to find a closed form for the following sequence:

$$a_0 = 1; \quad a_{n+1} = 2a_n + n$$

*Hint:* When doing partial fraction decomposition with a denominator of the form $(x-a)^2(x-b)$, you may need to express your expression as a sum of three fractions: $\frac{c}{(x-a)^2} + \frac{d}{x-a} + \frac{e}{x-b}$.''

# Part 3: Dice

**Definition 17:**
A *die* is a device that randomly selects a positive integer from a finite list of options. For example, the standard 6-sided die selects a value from $[1, 2, 3, 4, 5, 6]$. We may have many sides with the same value, as in $[1, 1, 2, 3]$.
To describe a die with a generating function, let $a_k$ be the number of times $k$ appears as a side of the die and consider $a_0 + a_1x + x_2x^2 + ....$
A die has a finite number of sides, so this will be a regular polynomial.

**Problem 18:**
What is the generating function of the standard 6-sided die?

**Problem 19:**
What is the generating function of the die with sides $[1, 2, 3, 5]$?

**Problem 20:**
Let $A(x)$ and $B(x)$ be the generating functions of two dice.
What is the significance of $A(1)$?

**Problem 21:**
Using formulas we found earlier, show that the $k^{\text{th}}$ coefficient of $A(x)B(x)$ is the number of ways to roll $k$ as the sum of the two dice.

**Problem 22:**
Find a generating function for the sequence $c_0, c_1, ...$, where $c_k$ is the probability that the sum of the two dice is $k$.

**Problem 23:**
Using generating functions, find two six-sided dice whose sum has the same distribution as the sum of two standard six-sided dice.
That is, for any integer $k$, the number if ways that the sum of the two nonstandard dice rolls as $k$ is equal to the number of ways the sum of two standard dice rolls as $k$. *Hint:* factor polynomials.

# Part 4: Coins

Consider the following problem:
"How many different ways can you make change for \$0.50
using pennies, nickels, dimes, quarters and half-dollars?"

Most ways of solving this involve awkward brute-force approaches that don't reveal anything interesting about the problem: how can we change our answer if we want to make change for \$0.51, or \$1.05, or some other quantity?

We can use generating functions to solve this problem in a general way.

**Definition 24:**
Let $p_0, p_1, p_2, ...$ be such that $p_k$ is the number of ways to make change for $k$ cents with only pennies. Similarly, let...

- $n_k$ be the number of ways to make change for $k$ cents with only nickels;
- $d_k$ be the number of ways using only dimes;
- $q_k$ be the number of ways using only quarters;
- and $h_k$ be the number of ways using only half-dollars.

**Problem 25:**
Let $p(x)$ be the generating function that corresponds to $p_n$.
Express $p(x)$ as a rational function.

**Problem 26:**
Modify **??** to find expressions for $n(x)$, $d(x)$, $q(x)$, and $h(x)$.

**Definition 27:**
Now, let $N(x)$ be the generating function for the sequence $n_0, n_1, ...$, where $n_k$ is the number of ways to make change for $k$ cents using pennies and nickels.
Similarly, let...

- let $D(x)$ be the generating function for the sequence using pennies, nickels, and dimes;
- let $Q(x)$ use pennies, nickels, dimes, and quarters;
- and let $H(x)$ use all coins.

**Problem 28:**
Express $N(x)$ as a rational function.

**Problem 29:**
Using the previous problem, write $D(x)$, then $Q(x)$, then $H(x)$ as rational functions.

**Problem 30:**
Using these generating functions, find recurrence relations for the sequences $N_k$, $D_k$, $Q_k$, and $H_k$.
*Hint:* Your recurrence relation for $N_k$ should refer to the previous values of itself and some values of $p_k$. Your recurrence for $D_k$ should refer to itself and $N_k$; the one for $Q_k$ should refer to itself $D_k$; and the one for $H_k$ should refer to itself and $Q_k$.

**Problem 31:**
Using these recurrence relations, fill following table and solve the original problem.

| $n$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_k$ | | | | | | | | | | | |
| $N_k$ | | | | | | | | | | | |
| $D_k$ | | | | | | | | | | | |
| $Q_k$ | | | | | | | | | | | |
| $H_k$ | | | | | | | | | | | |

# Part 5: Extra Problems

**Problem 32: USAMO 1996 Problem 6**

Determine (with proof) whether there is a subset $X$ of the nonnegative integers with the following property: for any nonnegative integer $n$ there is exactly one solution of $a + 2b = n$ with $a, b \in X$. (The original USAMO question asked about all integers, not just nonnegative - this is harder, but still approachable with generating functions.)

**Problem 33: IMO Shortlist 1998**

Let $a_0, a_1, ...$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where $i, j, k$ are not necessarily distinct. Determine $a_1998$.

**Problem 34: USAMO 1986 Problem 5**

By a partition $\pi$ of an integer $n \geq 1$, we mean here a representation of $n$ as a sum of one or more positive integers where the summands must be put in nondecreasing order. (e.g., if $n = 4$, then the partitions $\pi$ are $1+1+1+1$, $1+1+2$, $1+3, 2+2$, and 4).

For any partition $\pi$, define $A(\pi)$ to be the number of ones which appear in $\pi$, and define $B(\pi)$ to be the number of distinct integers which appear in $\pi$ (e.g, if $n = 13$ and $\pi$ is the partition $1+1+2+2+2+5$, then $A(\pi) = 2$ and $B(\pi) = 3$).

Show that for any fixed $n$, the sum of $A(\pi)$ over all partitions of $\pi$ of $n$ is equal to the sum of $B(\pi)$ over all partitions of $\pi$ of $n$.

**Problem 35: USAMO 2017 Problem 2**

Let $m_1, m_2, ..., m_n$ be a collection of $n$ distinct positive integers. For any sequence of integers $A = (a_1, ..., a_n)$ and any permutation $w = w_1, ..., w_n$ of $m_1, ..., m_n$, define an $A$-inversion of $w$ to be a pair of entries $w_i, w_j$ with $i < j$ for which one of the following conditions holds:

- $ai \geq wi > wj$
- $wj > ai \geq wi$
- $wi > wj > ai$

Show that for any two sequences of integers $A = (a_1, ..., a_n)$ and $B = (b_1, ..., b_n)$ and for any positive integer $k$, the number of permutations of $m_1, ..., m_n$ having exactly $k$ $A$-inversions is equal to the number of permutations of $m_1, ..., m_n$ having exactly $k$ $B$-inversions. (The original USAMO problem allowed the numbers $m_1, ..., m_n$ to not necessarily be distinct.)