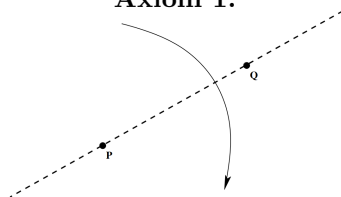


# Origami

Prepared by everyone on February 5, 2025

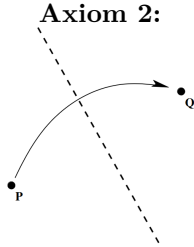
## Part 1: Axioms of Origami

**Axiom 1:**



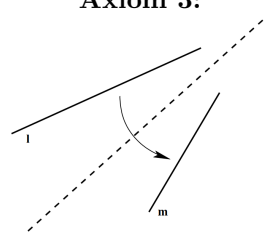
Given two points, we can fold a line between them.

**Axiom 2:**



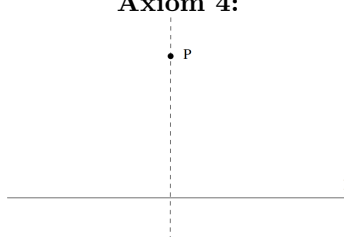
Given two points, we can make a fold that places one atop the other.

**Axiom 3:**



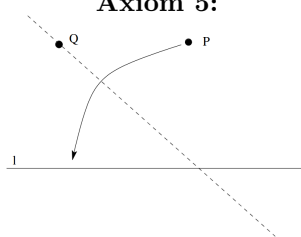
Given two lines, we can make a fold that places one atop the other.

**Axiom 4:**



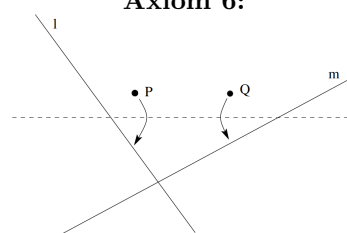
Given a point and a line, we can make a fold through the point and perpendicular to the line.

**Axiom 5:**



Given two points and a line, we can make a fold through one point that places the second on the line.

**Axiom 6:**

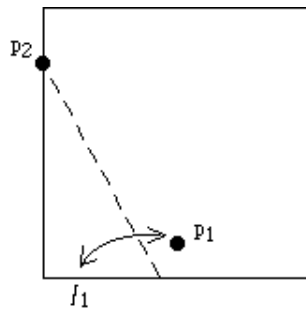


Given two points and two lines, we can make a fold that places each point on a line.

**Problem 1:**

Proposed by Nikita

- a:** Take a piece of paper. Let the bottom edge be  $l_1$  and take  $p_1$  to be a point in the middle and close to  $l_1$ . Then choose  $p_2$  to be anywhere on the left or right edge of the square and perform Axiom 5. Then choose a different  $p_2$ . Repeat this 8 or 9 times keeping the same  $p_1$  and choosing different  $p_2$ 's. What do you see?



- b:** Then, take another piece of paper. Draw two random intersecting lines  $l_1$  and  $l_2$  and points  $p_1$  and  $p_2$  about an inch close to their intersection. Perform a Beloch fold for them.

**Problem 2:**

Proposed by James

- a:** Given a circle, its center, and a point  $p$  on the circle, use origami to construct a tangent line to the circle that passes through  $p$ .
- b:** Given a circle, its center, and a point  $p$  on the circle, use origami to construct an equilateral triangle inscribed in the circle that passes through  $p$ .
- c:** Given a triangle, use origami to construct the center of the circle inscribed in it and its tangent points.

**Problem 3:**

Proposed by Nikita

Use origami to find the other three notable points in the given triangle: circumcenter, centroid and orthocenter.

**Problem 4:**

Proposed by Nikita

- a:** Emulate Axiom 5 with a compass and straightedge.
- b:** In your emulation, probably, there is a choice of which of the two intersections of a circle and a line to take. Does it mean that there are two ways to perform the fold?

**Problem 5:**

Proposed by Nikita

Prove that  $\sqrt[3]{2} \neq \frac{a}{b}$  for any  $a, b \in \mathbb{N}$ .

**Problem 6:**

Proposed by Nikita

- a:** Construct a regular hexagon using a ruler and compass.
- b:** Cut the triangle with angles  $72^\circ$ ,  $72^\circ$ ,  $36^\circ$  into 2 isosceles triangles.
- c:** Using triangle similarity, prove that the ratio of the sides in this triangle is equal to the golden ratio  $\varphi = \frac{1+\sqrt{5}}{2}$ .
- d:** Find a way to construct a regular pentagon using only a ruler and compass.

**Problem 7:**

Proposed by Nikita

- a:** Use origami to divide a given segment into 3 equal parts.
- b:** Use origami to divide a given segment into  $n$  equal parts.

**Problem 8:**

Proposed by ?

- a:** In the lecture, you saw that Axioms 1–5 are all able to be simulated by compass and straightedge constructions. Is the following claim a correct *deduction* from the above? (In other words, does simulating Axioms 1–5 *prove* the claim?)  
Claim: “In all cases, origami constructions are at least as powerful as compass and straightedge constructions.”
- b:** Is the claim true? Argue *both* sides with yourself (or with a classmate).
- c:** (Hard) Prove the sense in which the claim is true. (Hint: recall from the lecture that all constructible lengths with straightedge and compass are rational, or of the form  $a + b\sqrt{c}$  with  $a, b, c$  rational, or of the form  $d + e\sqrt{f}$  with  $d, e, f$  of the form  $a + b\sqrt{c}$  with  $a, b, c$  rational, etc.)

**Problem 9:**

Proposed by Mark

Do each of the following with a compass and ruler.

Do not use folds.

- a:** Divide a circle into five parts of equal area.
- b:** Divide a circle into seven parts of equal area.
- c:** Divide a circle into  $n$  parts of equal area.

**Problem 10:**

Proposed by Sunny

Using a compass and ruler, find two circles tangent to a point  $D$  and lines  $AB$  and  $AC$ . (Problem of Apollonius, PLL case)

*Hint:* All circles tangent to  $AB$  and  $AC$  are homothetic with centre at  $A$ . What does this mean? Also, the angle bisector may help.

