

# Definable Sets

Prepared by Mark on February 5, 2025

## Part 1: Logical Algebra

### Definition 1:

*Logical operators* operate on the values `{true, false}`, just like algebraic operators operate on numbers.

In this handout, we'll use the following operators:

- $\neg$ : not
- $\wedge$ : and
- $\vee$ : or
- $\rightarrow$ : implies
- $()$ : parenthesis.

The function of these is defined by *truth tables*:

and			or			implies			not	
$A$	$B$	$A \wedge B$	$A$	$B$	$A \vee B$	$A$	$B$	$A \rightarrow B$	$A$	$\neg A$
F	F	F	F	F	F	F	F	T	T	F
F	T	F	F	T	T	F	T	T	F	T
T	F	F	T	F	T	T	F	F		
T	T	T	T	T	T	T	T	T		

$A \wedge B$  is **true** only if both  $A$  and  $B$  are **true**.  $A \vee B$  is **true** if  $A$  or  $B$  (or both) are **true**.

$\neg A$  is the opposite of  $A$ , which is why it looks like a “negative” sign.

$A \rightarrow B$  is a bit harder to understand. Read aloud, this is “ $A$  implies  $B$ .”

The only time  $\rightarrow$  produces **false** is when **true**  $\rightarrow$  **false**. This fact may seem counterintuitive, but will make more sense as we progress through this handout.

*Hint:* Think about it—if event  $\alpha$  implies  $\beta$ , it is impossible for  $\alpha$  to occur without  $\beta$ .

This is the only impossibility. All other variants are valid.

### Problem 2:

Evaluate the following.

- $\neg T$
- $F \vee T$
- $T \wedge T$
- $(T \wedge F) \vee T$
- $(\neg(F \vee \neg T)) \rightarrow \neg T$
- $(F \rightarrow T) \rightarrow (\neg F \vee \neg T)$

**Problem 3:**

Evaluate the following.

- $A \rightarrow \text{T}$  for any  $A$
- $(\neg(A \rightarrow B)) \rightarrow A$  for any  $A, B$
- $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$  for any  $A, B$

**Problem 4:**

Show that  $\neg(A \rightarrow \neg B)$  is equivalent to  $A \wedge B$ .

That is, show that these expressions always evaluate to the same value given the same  $A$  and  $B$ .

*Hint:* Use a truth table

**Problem 5:**

Write an expression equivalent to  $A \vee B$  using only  $\neg$ ,  $\rightarrow$ , and  $()$ ?

Note that both  $\wedge$  and  $\vee$  can be defined using the other logical symbols.

The only logical symbols we *need* are  $\neg$ ,  $\rightarrow$ , and  $()$ .

We include  $\wedge$  and  $\vee$  to simplify our expressions.

## Part 2: Structures

### Definition 6:

A *universe* is a set of meaningless objects. Here are a few examples:

- $\{a, b, \dots, z\}$
- $\{0, 1\}$
- $\mathbb{Z}, \mathbb{R}$ , etc.

### Definition 7:

A *structure* consists of a universe and a set of *symbols*.

A structure's symbols give meaning to the objects in its universe.

Symbols come in three types:

- *Constant symbols*, which let us specify specific elements of our universe.  
Examples:  $0, 1, \frac{1}{2}, \pi$
- *Function symbols*, which let us navigate between elements of our universe.  
Examples:  $+, \times, \sin x, \sqrt{x}$   
Note that symbols we usually call “operators” are functions under this definition.  
The only difference between  $a + b$  and  $+(a, b)$  is notation.
- *Relation symbols*, which let us compare elements of our universe.  
Examples:  $<, >, \leq, \geq$

The equality check  $=$  is *not* a relation symbol. It is included in every structure by default.

By definition,  $a = b$  is true if and only if  $a$  and  $b$  are the same element of our universe.

### Example 8:

The first structure we'll look at is the following:

$$\left( \mathbb{Z} \mid \{0, 1, +, -, <\} \right)$$

This is a structure over the universe  $\mathbb{Z}$  that provides the following symbols:

- Constants:  $\{0, 1\}$
- Functions:  $\{+, -\}$
- Relations:  $\{<\}$

If we look at our set of constant symbols, we see that the only integers we can directly refer to in this structure are 0 and 1. If we want any others, we must define them using the tools this structure offers.

To “define” an element of a set, we need to write a sentence that is only true for that element.

If we want to define 2 in the structure above, we could use the following sentence:

$$\text{“2 is the } x \text{ that satisfies } [1 + 1 = x].\text{”}$$

This is a valid definition because 2 is the *only* element of  $\mathbb{Z}$  for which  $[1 + 1 = x]$  evaluates to **true**.

### Problem 9:

Define  $-1$  in  $\left( \mathbb{Z} \mid \{0, 1, +, -, <\} \right)$ .

Let us formalize what we found in the previous two problems.

**Definition 10: Formulas**

A *formula* in a structure  $S$  is a well-formed string of constants, functions, relations, and logical operators.

You already know what a “well-formed string” is:  $1 + 1$  is fine,  $\sqrt{+}$  is nonsense.

For the sake of time, I will not provide a formal definition — it isn’t particularly interesting.

As a quick example, the formula  $\psi := [\neg(1 = 1)]$  is always false,

and  $\varphi(x) := [1 + 1 = x]$  evaluates to **true** only when  $x$  is 2.

**Definition 11: Free Variables**

A formula can contain one or more *free variables*. These are denoted  $\varphi(a, b, \dots)$ .

Formulas with free variables let us define “properties” that certain objects have.

For example, consider the two formulas from the previous definition,  $\psi$  and  $\varphi$ :

- $\psi := [\neg(1 = 1)]$   
There are no free variables in this formula.  
In any structure,  $\psi$  is always either **true** or **false**.
- $\varphi(x) := [1 + 1 = x]$   
This formula has one free variable, labeled  $x$ .  
The value of  $\varphi(x)$  depends on the  $x$  we’re talking about:  
 $\varphi(72)$  is false, and  $\varphi(2)$  is true.

This “free variable” notation is very similar to the function notation we are used to:

The values of both  $\varphi(x) := [x > 0]$  and  $f(x) = x + 1$  depend on  $x$ .

**Definition 12: Definable Elements**

Let  $S$  be a structure over a universe  $U$ .

We say an element  $x \in U$  is *definable in  $S$*  if we can write a formula  $\varphi(x)$  that only  $x$  satisfies.

**Problem 13:**

Define 2 in the structure  $(\mathbb{Z}^+ \mid \{4, \times\})$ .

*Hint:*  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ . Also,  $2 \times 2 = 4$ .

**Problem 14:**

Try to define 2 in the structure  $(\mathbb{Z} \mid \{4, \times\})$ .

Why can't you do it?

**Problem 15:**

Consider the structure  $(\mathbb{R}_0^+ \mid \{1, 2, \div\})$

- Define  $2^2$
- Define  $2^n$  for all positive integers  $n$
- Define  $2^{-n}$  for all positive integers  $n$
- What other numbers can we define in this structure?

*Hint:* There is at least one more “class” of numbers we can define.

## Part 3: Quantifiers

Recall the logical symbols we introduced earlier:  $()$ ,  $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\rightarrow$   
We will now add two more:  $\forall$  (for all) and  $\exists$  (exists).

### Definition 16:

$\forall$  and  $\exists$  are *quantifiers*. They allow us to make statements about arbitrary symbols.  
Quantifiers are aptly named: they tell us *how many* symbols satisfy a certain sentence.

Let's look at  $\forall$  first. If  $\varphi(x)$  is a formula,  
the formula  $\forall x \varphi(x)$  is true only if  $\varphi$  is true for all  $x$  in our universe.

For example, take the formula  $\forall x (0 < x)$ .

In English, this means "For any  $x$ ,  $x$  is bigger than zero," or simply "Any  $x$  is positive."

$\exists$  is very similar: the formula  $\exists x \varphi(x)$  is true if there is at least one  $x$  for which  $\varphi(x)$  is true.  
For example,  $\exists (0 < x)$  means "there is a positive number in our set."

### Problem 17:

Which of the following are true in  $\mathbb{Z}$ ? Which are true in  $\mathbb{R}_0^+$ ?

$\mathbb{R}_0^+$  is the set of positive real numbers and zero.

- $\forall x (x \geq 0)$
- $\neg(\exists x (x = 0))$
- $\forall x [\exists y (y \times y = x)]$
- $\forall xy \exists z (x < z < y)$       This is a compact way to write  $\forall x (\forall y (\exists z (x < z < y)))$
- $\neg \exists x (\forall y (x < y))$

**Problem 18:**

Does the order of  $\forall$  and  $\exists$  in a formula matter?

What's the difference between  $\exists x \forall y (x \leq y)$  and  $\forall y \exists x (x \leq y)$ ?

*Hint:* Consider  $\mathbb{R}^+$ , the set of positive reals. Zero is not positive.

Which of the above formulas is true in  $\mathbb{R}^+$  and which is false?

**Problem 19:**

Define 0 in  $(\mathbb{Z} \mid \{\times\})$

**Problem 20:**

Define 1 in  $(\mathbb{Z} \mid \{\times\})$

**Problem 21:**

Define  $-1$  in  $(\mathbb{Z} \mid \{0, <\})$

**Problem 22:**

Let  $\varphi(x)$  be a formula.

Write a formula equivalent to  $\forall x \varphi(x)$  using only logical symbols and  $\exists$ .



## Part 4: Definable Sets

Armed with  $()$ ,  $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\rightarrow$ ,  $\forall$ , and  $\exists$ , we have the tools to define sets.

### Definition 23: Set-Builder Notation

Say we have a sentence  $\varphi(x)$ .

The set of all elements that satisfy that sentence may be written as follows:

$$\{x \mid \varphi(x)\}$$

This is read “The set of  $x$  where  $\varphi$  is true” or “The set of  $x$  that satisfy  $\varphi$ .”

For example, take the formula  $\varphi(x) = \exists y (y + y = x)$ .

The set of all even integers can then be written as

$$\{x \mid \exists y (y + y = x)\}$$

### Definition 24: Definable Sets

Let  $S$  be a structure with a universe  $U$ .

We say a subset  $M$  of  $U$  is *definable* if we can write a formula that is true for some  $x$  if and only if  $M$  contains  $x$ .

For example, consider the structure  $(\mathbb{Z} \mid \{+\})$ .

Only even numbers satisfy the formula  $\varphi(x) := [\exists y (y + y = x)]$ ,

so we can define “the set of even numbers” as  $\{x \mid \exists y (y + y = x)\}$ .

Remember—we can only use symbols that are available in our structure!

### Problem 25:

The empty set is definable in any structure. How?

### Problem 26:

Define  $\{0, 1\}$  in  $(\mathbb{Z}_0^+ \mid \{<\})$  *Hint:* Define 0 and 1 as elements first, and remember that we can use logical symbols.

### Problem 27:

Define the set of prime numbers in  $(\mathbb{Z} \mid \{\times, \div, <\})$ .

*Hint:* A prime number is an integer that is positive and is only divisible by 1 and itself.

**Problem 28:**

Define  $\mathbb{R}_0^+$  in  $(\mathbb{R} \mid \{\times\})$

**Problem 29:**

Let  $\triangle$  be a relational symbol.  $a \triangle b$  is only true if  $a$  divides  $b$ .

Define the set of prime numbers in  $(\mathbb{Z}^+ \mid \{\triangle\})$

**Theorem 30: Lagrange's Four Square Theorem**

Every natural number may be written as a sum of four integer squares.

**Problem 31:**

Define  $\mathbb{Z}_0^+$  in  $(\mathbb{Z} \mid \{\times, +\})$

**Problem 32:**

Define  $<$  in  $(\mathbb{Z} \mid \{\times, +\})$

*Hint:* We can't formally define a relation yet. Don't worry about that for now.

You can rephrase this question as “given  $x, y \in \mathbb{Z}$ , write a formula  $\varphi(x, y)$  that is only true if  $x < y$ ”

**Problem 33:**

Consider the structure  $S = (\mathbb{R} \mid \{0, \diamond\})$

The relation  $a \diamond b$  holds if  $|a - b| = 1$

**Part 1:**

Define  $\{-1, 1\}$  in  $S$ .

**Part 2:**

Define  $\{-2, 2\}$  in  $S$ .

**Problem 34:**

Let  $\mathcal{P}$  be the set of all subsets of  $\mathbb{Z}_0^+$ . This is called the *power set* of  $\mathbb{Z}_0^+$ .

Let  $S$  be the structure  $(\mathcal{P} \mid \{\subseteq\})$

**Part 1:**

Show that the empty set is definable in  $S$ .

*Hint:* Defining  $\{\}$  with  $\{x \mid \neg x = x\}$  is **not** what we need here.

We need  $\emptyset \in \mathcal{P}$ , the “empty set” element in the power set of  $\mathbb{Z}_0^+$ .

**Part 2:**

Let  $x \approx y$  be a relation on  $\mathcal{P}$ .  $x \approx y$  holds if  $x \cap y \neq \{\}$ .

Show that  $\approx$  is definable in  $S$ .

**Part 3:**

Let  $f$  be the function on  $\mathcal{P}$  defined by  $f(x) = \mathbb{Z}_0^+ - x$ . This is called the *complement* of  $x$ .

Show that  $f$  is definable in  $S$ .

*Hint:* You can define a function by writing a formula  $\varphi(x, y)$  that is only true when  $y = f(x)$ .

## Part 5: Equivalence

### Notation:

Let  $S$  be a structure and  $\varphi$  a formula.

If  $\varphi$  is true in  $S$ , we write  $S \models \varphi$ .

This is read “ $S$  satisfies  $\varphi$ ”

### Definition 35:

Let  $S$  and  $T$  be structures.

We say  $S$  and  $T$  are *equivalent* (and write  $S \equiv T$ ) if for any formula  $\varphi$ ,  $S \models \varphi \iff T \models \varphi$ .

If  $S$  and  $T$  are not equivalent, we write  $S \not\equiv T$ .

### Problem 36:

Show that  $(\mathbb{Z} \mid \{+, 0\}) \not\equiv (\mathbb{R} \mid \{+, 0\})$

### Problem 37:

Show that  $(\mathbb{Z} \mid \{+, 0\}) \not\equiv (\mathbb{N} \mid \{+, 0\})$

### Problem 38:

Show that  $(\mathbb{R} \mid \{+, 0\}) \not\equiv (\mathbb{N} \mid \{+, 0\})$

### Problem 39:

Show that  $(\mathbb{R} \mid \{+, 0\}) \not\equiv (\mathbb{Z}^2 \mid \{+, 0\})$

### Problem 40:

Show that  $(\mathbb{Z} \mid \{+, 0\}) \not\equiv (\mathbb{Z}^2 \mid \{+, 0\})$