

Wallpaper Symmetry

Prepared by Mark on May 9, 2025

Instructor's Handout

This handout contains solutions and notes.
Recompile without solutions before distributing.

Section 1: Wallpaper Symmetries

Definition 1:

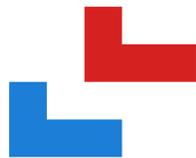
A *Euclidean isometry* is a transformation of the plane that preserves distances. Intuitively, an isometry moves objects on the plane without deforming them.

There are four classes of Euclidean isometries:

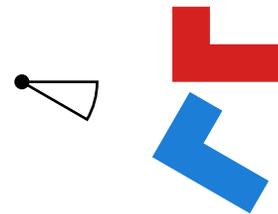
- translations
- reflections
- rotations
- glide reflections

We can prove there are no others, but this is beyond the scope of this handout.

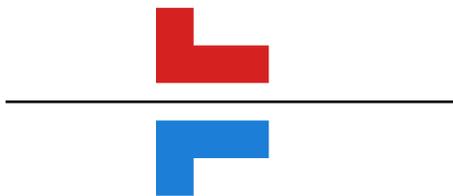
A simple example of each isometry is shown below:



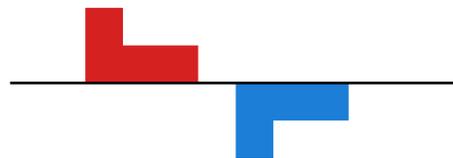
Translation



Rotation



Reflection



Glide reflection

Definition 2:

A *wallpaper* is a two-dimensional pattern that...

- has translational symmetry in at least two non-parallel directions (and therefore fills the plane)
“Translational symmetry” means that we can slide the entire wallpaper in some direction, eventually mapping the pattern to itself.
- has a countable number of reflection, rotation, or glide symmetries.

Problem 3:

Is a plain square grid a valid wallpaper?

Solution:

Yes!

- It has translational symmetry in the horizontal and vertical directions
- It has a countable number of symmetries—namely, six distinct mirror lines (horizontal, vertical, and diagonal) duplicated once per square.
- A square grid is *442

Problem 4:

Is the empty plane a valid wallpaper?

Solution:

No, since it has uncountably many symmetries.

Section 2: Mirror Symmetry

Definition 5:

A *reflection* is a transformation of the plane obtained by reflecting all points about a line.

If this reflection maps the wallpaper to itself, we have a *mirror symmetry*.

If n such mirror lines intersect at a point, they form a *mirror node of order n* .

Mirror nodes with order 1 do not exist (i.e, $n \geq 2$). A line does not intersect itself!

Two mirror nodes on a wallpaper are identical if we can map one to the other with a translation and a rotation while preserving the pattern on that wallpaper.

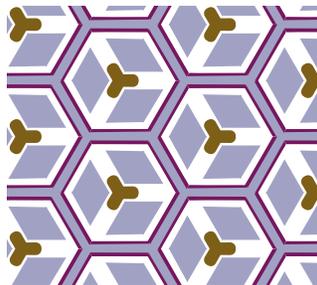
Problem 6:

Find all three distinct mirror nodes in the following pattern.

What is the order of each node?

Hint: You may notice rotational symmetry in this pattern.

Don't worry about that yet, we'll discuss it later.



Solution:

The mirror nodes are:

- the center of the amber cross
- the center of each right-handed group of three adjacent hexagons
- the center of each left-handed group of three adjacent hexagons

Definition 7:

Orbifold notation gives us a way to describe the symmetries of a wallpaper.

It defines a *signature* that fully describes all the symmetries of a given pattern.

We will introduce orbifold notation one symmetry at a time.

Definition 8:

In orbifold notation, mirror nodes are denoted by a $*$ followed by a list of integer.

Every integer n following a $*$ denotes a mirror node of order n .

The order of these integers doesn't matter. $*234$ and $*423$ are the same signature.

However, we usually denote n -fold symmetries in descending order (that is, like $*432$).

If we have many nodes of the same order, integers may be repeated.

Problem 9:

What is the signature of the wallpaper in Problem 6?

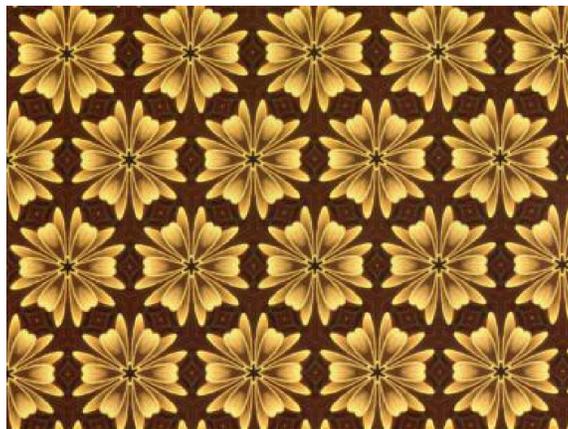
Hint: Again, ignore rotational symmetry for now.

Solution:

It is $*333$

Problem 10:

Find the signature of the following pattern.



Solution:

It is *632:

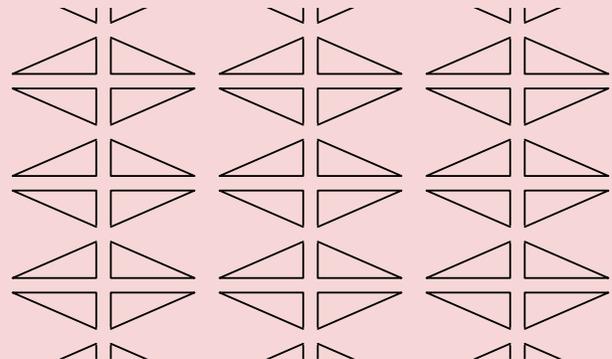
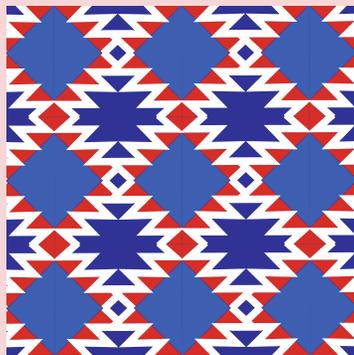


Problem 11:

Draw a wallpaper pattern with signature *2222

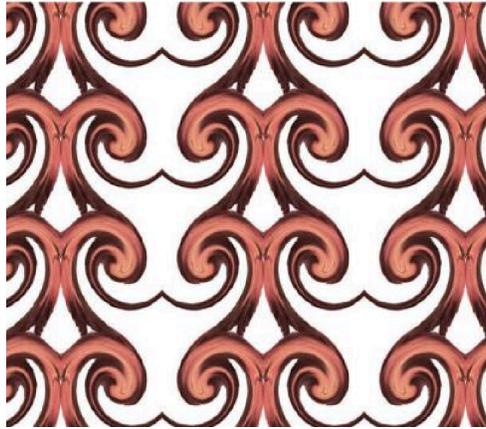
Solution:

Sample solutions are below.



Remark 12:

In an exceptional case, we have two parallel mirror lines.
Consider the following pattern:



The signature of this pattern is **

Problem 13:

Draw another wallpaper pattern with signature **.

Section 3: Rotational Symmetry

Definition 14:

A wallpaper may also have n -fold rotational symmetry about a point.

This means there are no more than n rotations around that point that map the wallpaper to itself.

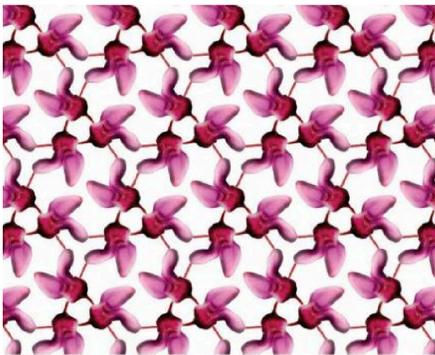
As before, two points of rotational symmetry are identical if we can perform a translation and rotation that maps one to the other without changing the wallpaper.

Definition 15:

In orbifold notation, rotation is specified similarly to reflection, but uses the prefix \diamond .

For example:

- $\diamond 333$ denotes a pattern with three distinct centers of rotation of order 3.
- $\diamond 4*2$ denotes a pattern with one rotation center of order 4 and one mirror node of order 2.



Problem 16:

Find the three rotation centers in the left wallpaper.
What are their orders?

Solution:

This is $\diamond 333$

Problem 17:

Find the signature of the pattern on the right.

Solution:

This is $\diamond 3*3$

Remark 18:

You may have noticed that we could have an ambiguous classification, since two reflections are equivalent to a translation and a rotation. We thus make the following distinction: *rotational symmetry that can be explained by reflection is not rotational symmetry.*

In other words, when classifying a pattern...

- we first find all mirror symmetries,
- then all rotational symmetries that are not accounted for by reflection.

Section 4: Glide Reflections

Definition 19:

Another type of symmetry is the *glide reflection*, denoted \times .

A glide reflection is the result of a translation along a line followed by reflection about that line.

For example, consider the following pattern:



Problem 20:

Convince yourself that all mirror lines in this pattern are *not* distinct. / In other words, this pattern has only one mirror symmetry.

Solution:

There may seem to be two, but they are identical.
We can translate one onto the other.

Problem 21:

Use the following picture to find the glide reflection in the above pattern.



Remark 22:

The signature of this wallpaper is $*\times$.

Definition 23:

If none of the above symmetries appear in a pattern, then we only have simple translational symmetry. We denote this with the signature \circ .

Remark 24:

In summary, to find the signature of a pattern:

- find the mirror lines ($*$) and the distinct intersections;
- then find the rotation centers (\diamond) not explained by reflection;
- then find all glide reflections (\times) that do not cross a mirror line.
- If we have none of the above, our pattern must be \circ .

Problem 25:

Find the signature of the following pattern:

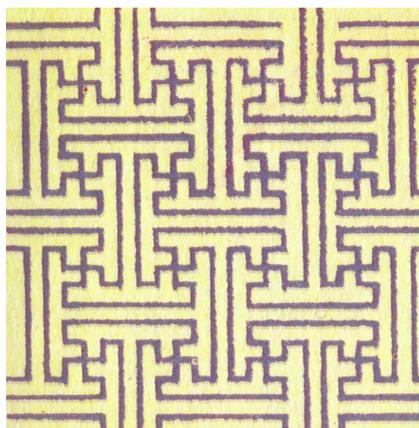


Solution:

This is $*\times$.

Problem 26:

Find the signature of the following pattern:



Solution:

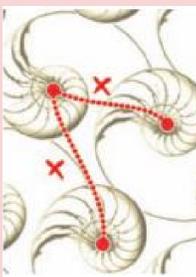
This is $\diamond 4*2$

Problem 27:

Find two glide reflections in the following pattern.
(and thus show that its signature is $\times\times$.)



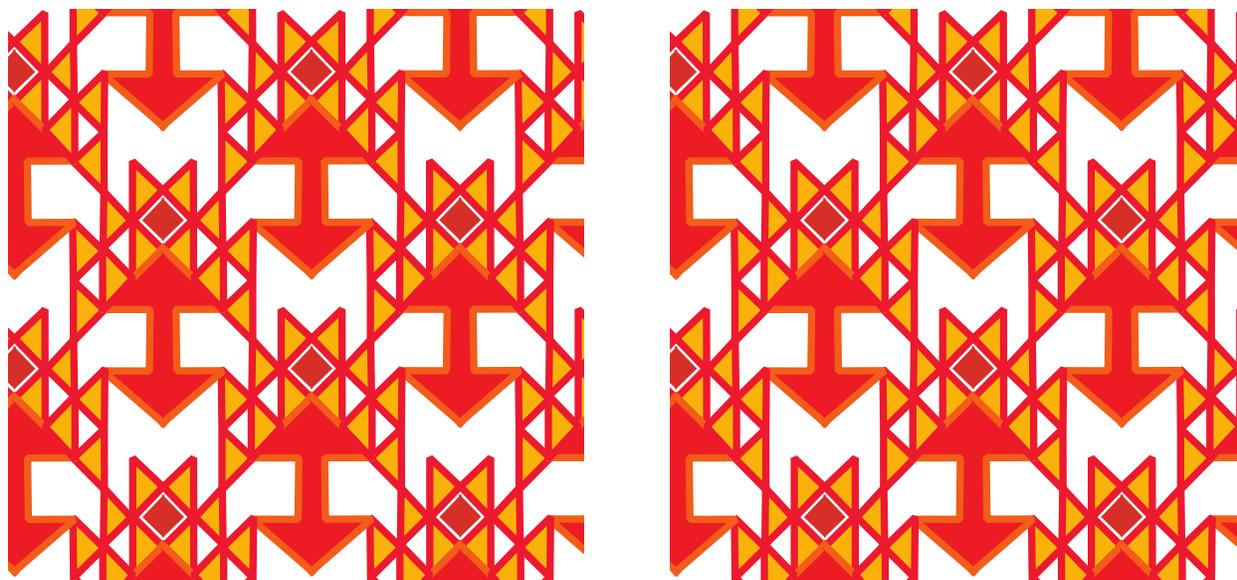
Solution:



Section 5: A few problems

Find the signatures of the following patterns. Mark all mirror nodes, rotation centers, and glide reflections. Each pattern is provided twice for convenience.

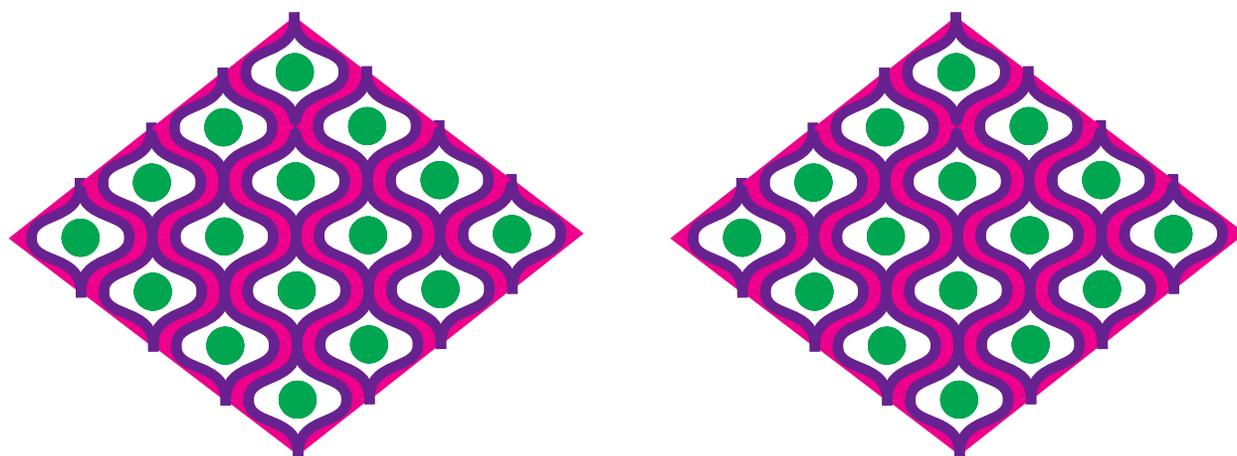
Problem 28:



Solution:

\times^*

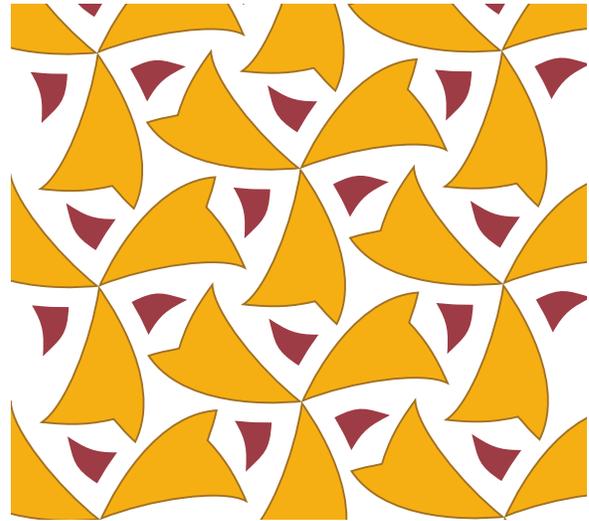
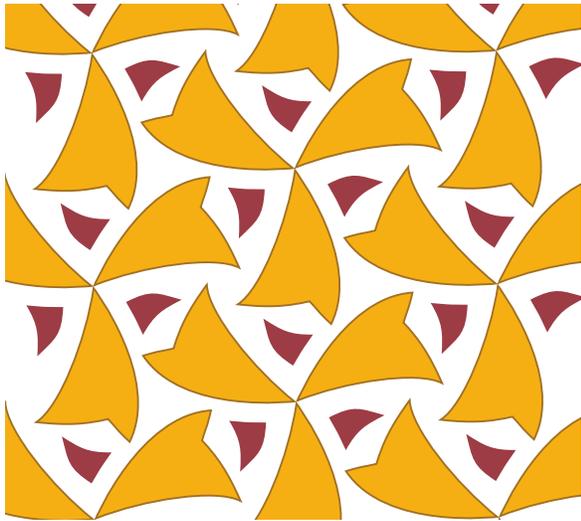
Problem 29:



Solution:

$\diamond 2^*22$

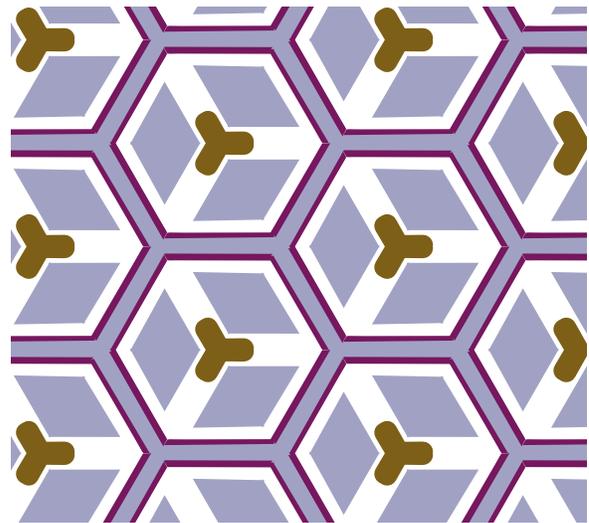
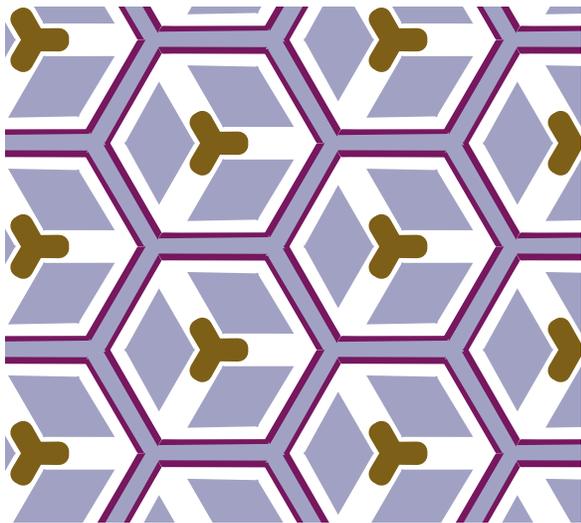
Problem 30:



Solution:

◇333

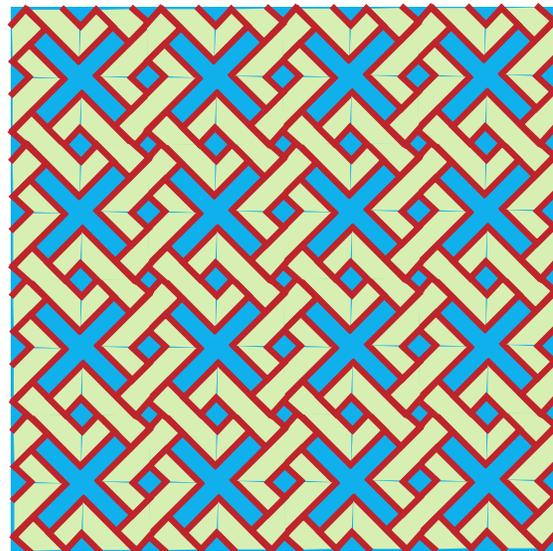
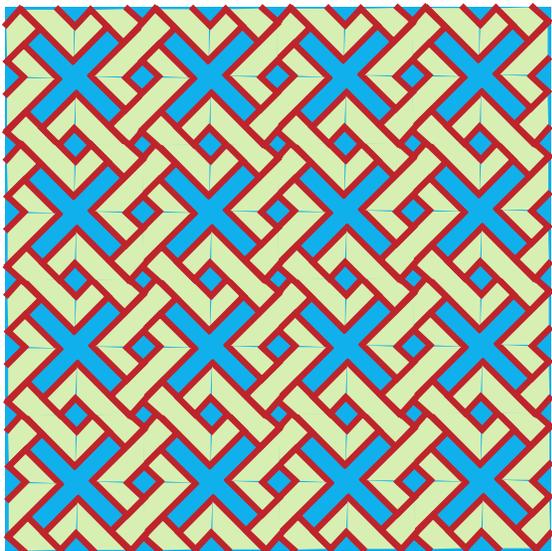
Problem 31:



Solution:

*333

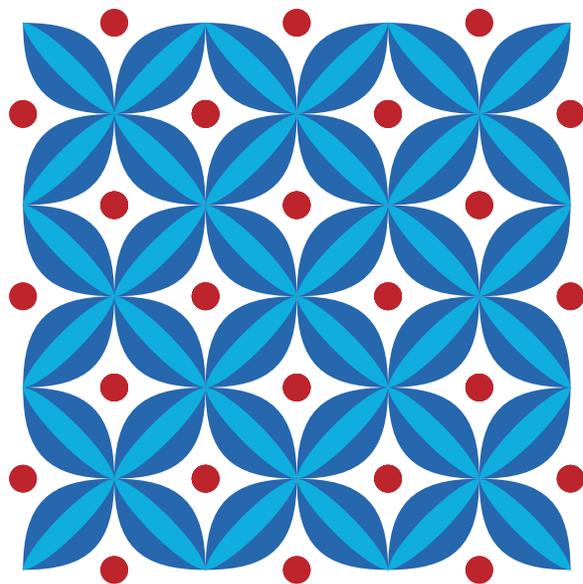
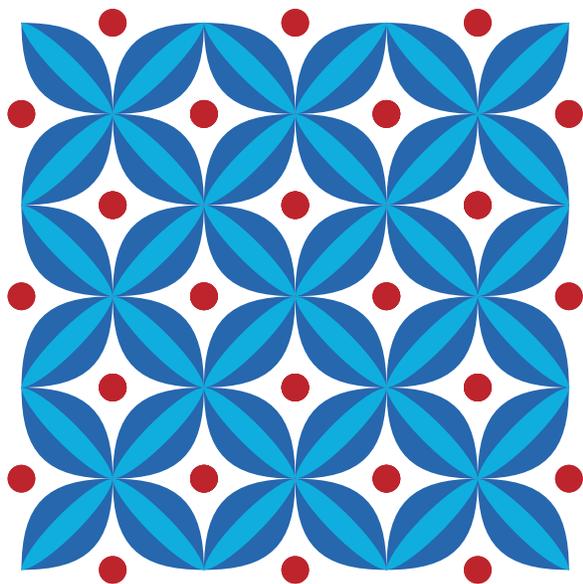
Problem 32:



Solution:

◇442

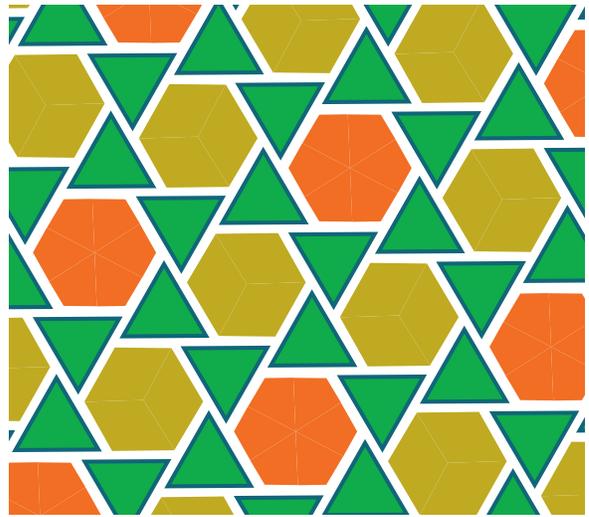
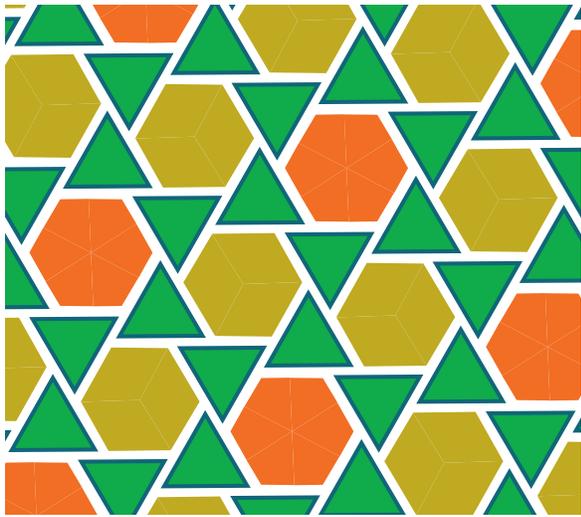
Problem 33:



Solution:

*442

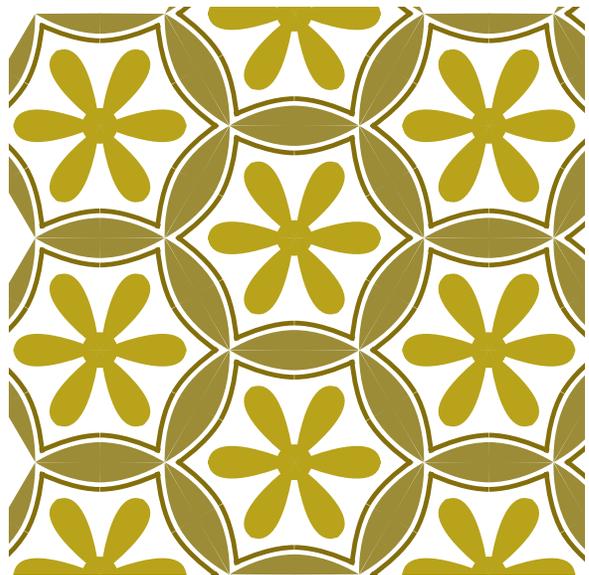
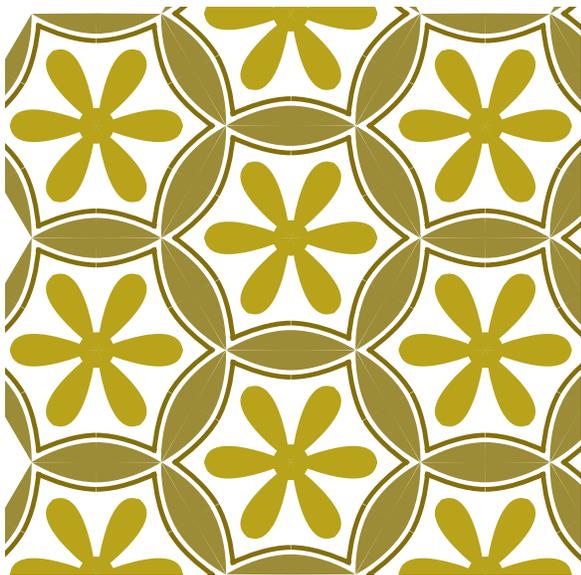
Problem 34:



Solution:

◇632

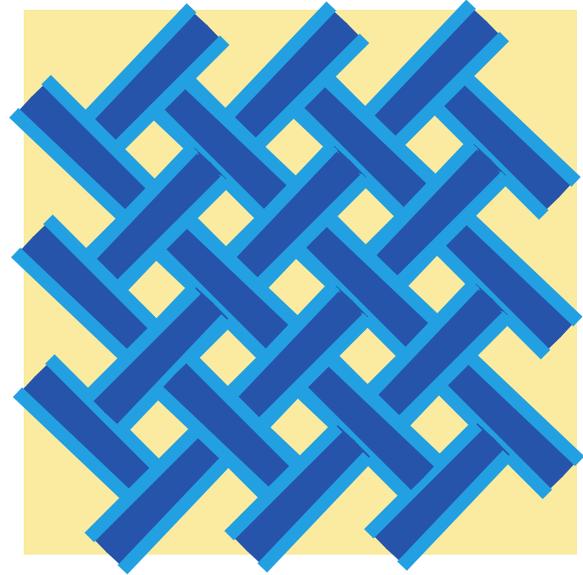
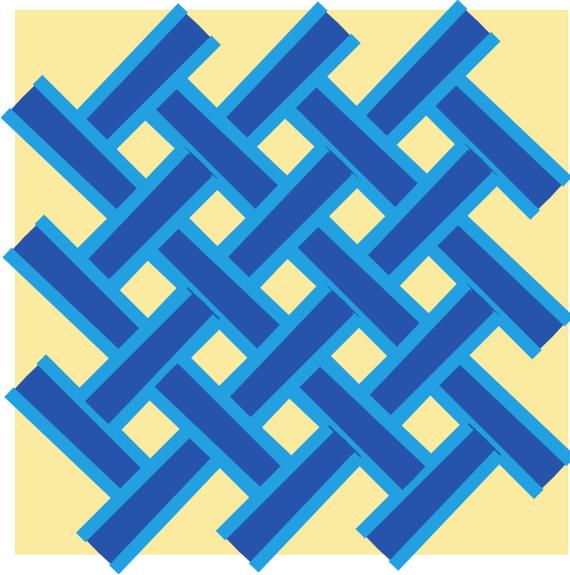
Problem 35:



Solution:

*632

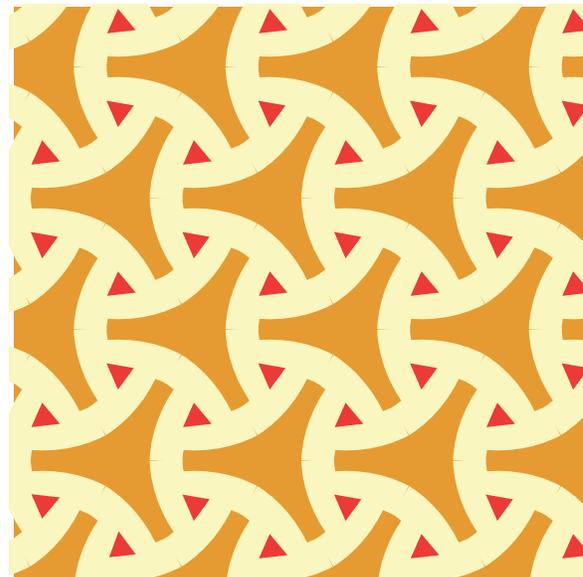
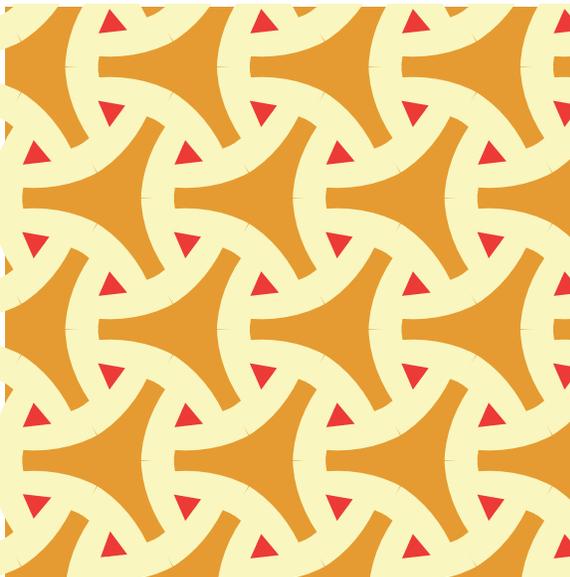
Problem 36:



Solution:

◇4*2

Problem 37:



Solution:

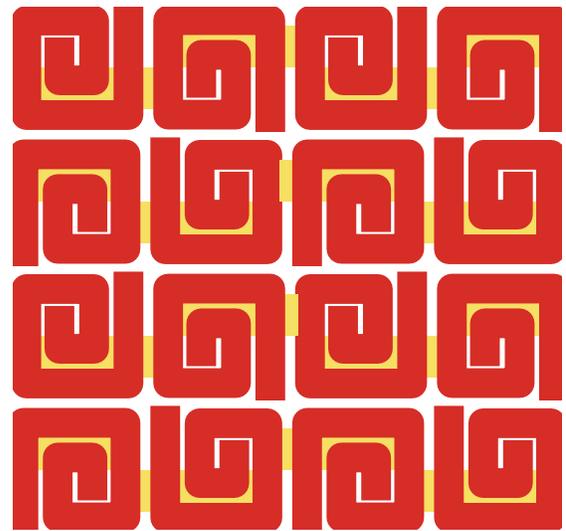
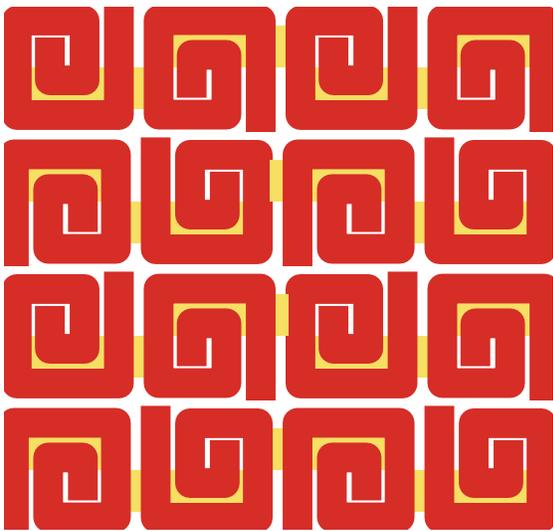
◇3*3

Problem 38:

Draw a wallpaper with the signature $*442$

Make sure there are no other symmetries!

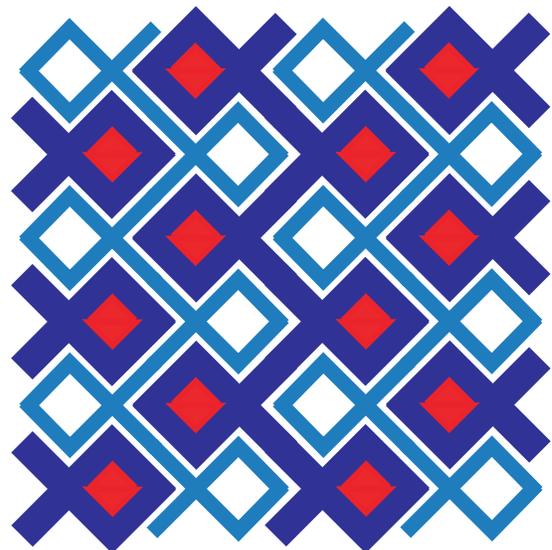
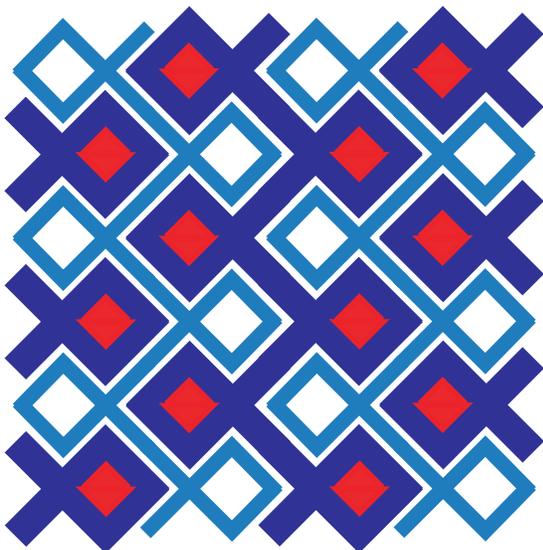
Problem 39:



Solution:

◇22×

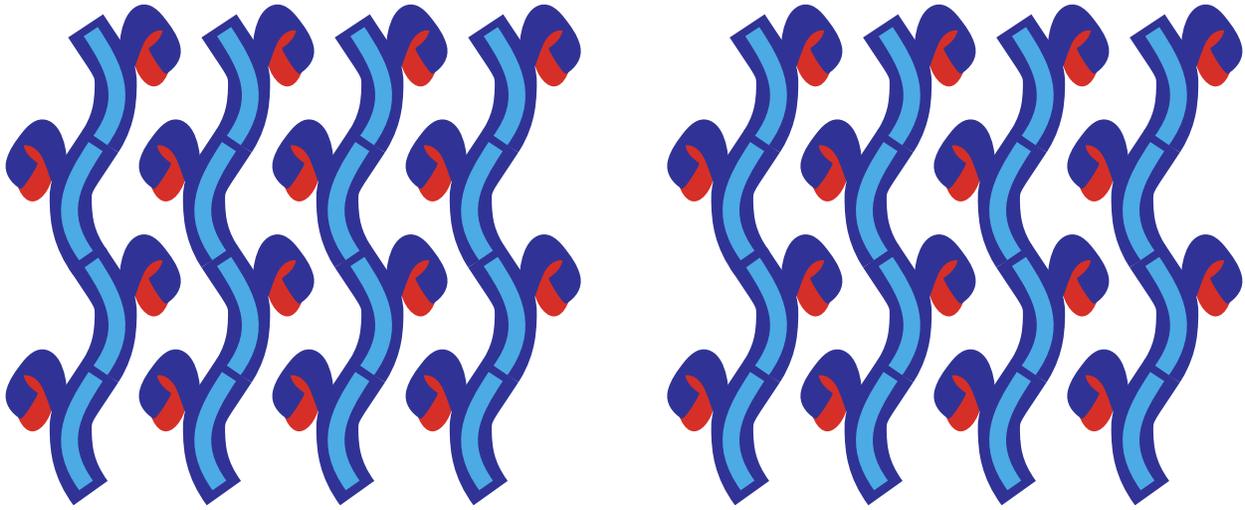
Problem 40:



Solution:

◇22*

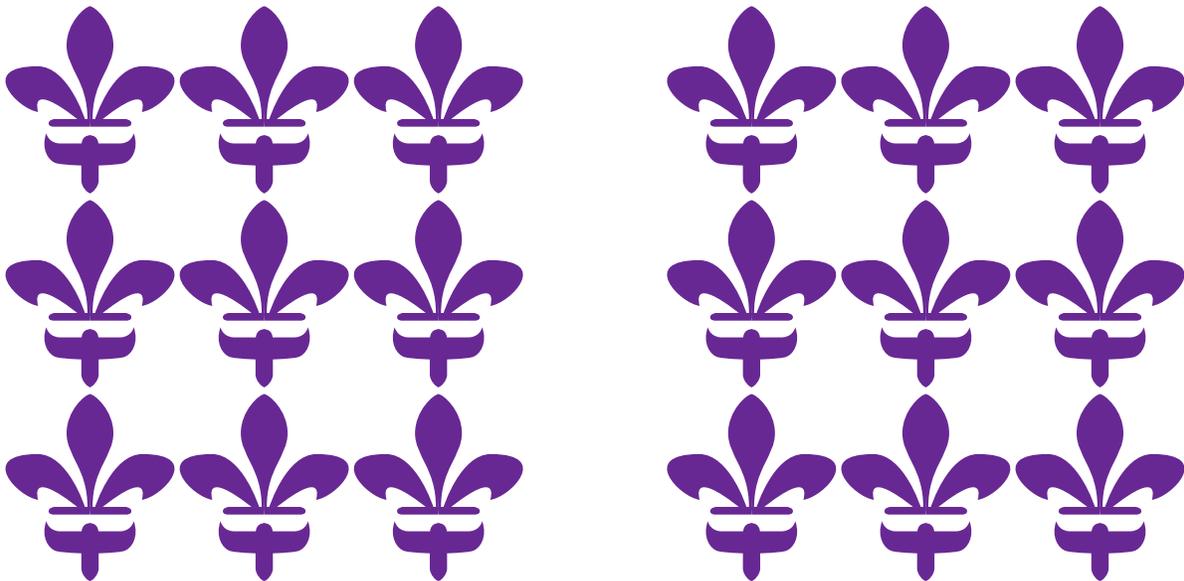
Problem 41:



Solution:

××

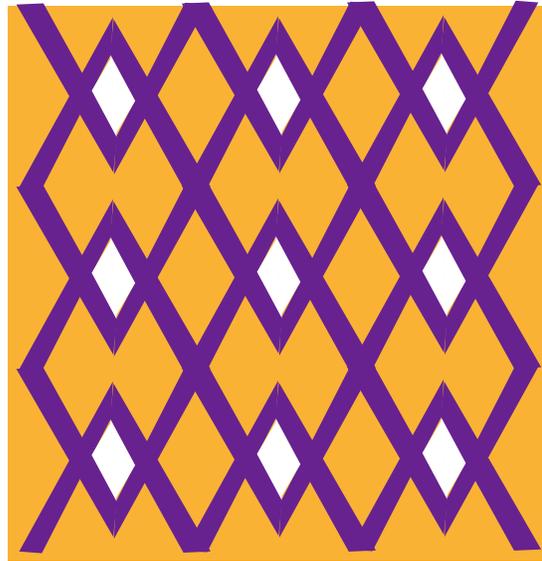
Problem 42:



Solution:

**

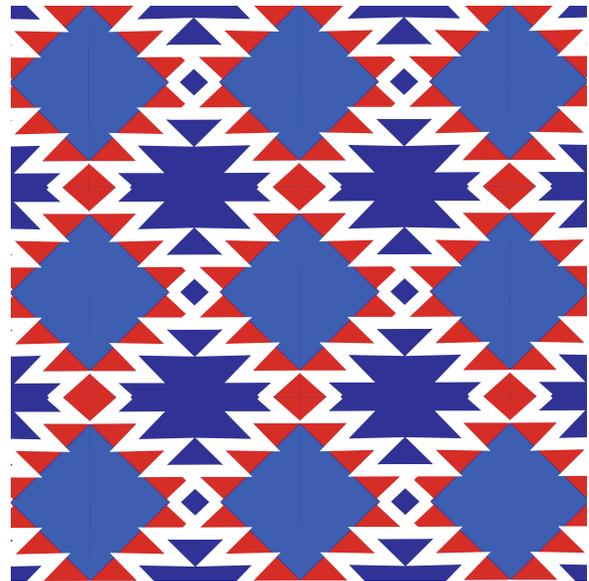
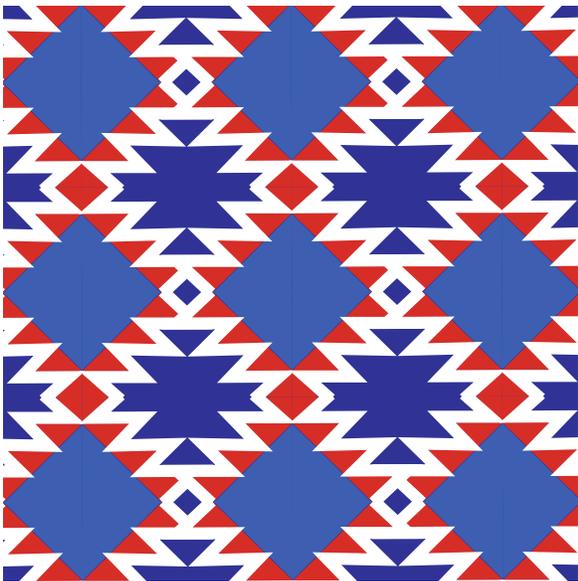
Problem 43:



Solution:

◇2222

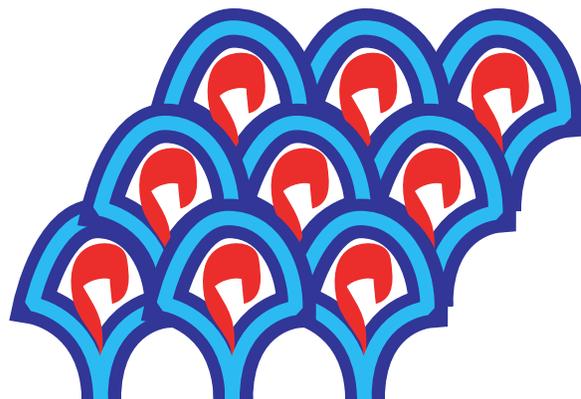
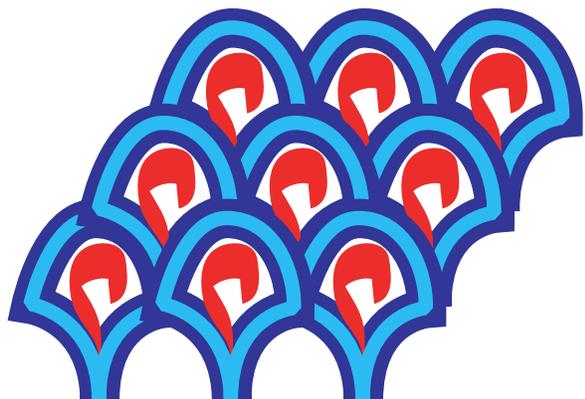
Problem 44:



Solution:

*2222

Problem 45:



Solution:

o

Section 6: The Signature-Cost Theorem

Definition 46:

First, we'll associate a *cost* to each type of symmetry in orbifold notation:

Symbol	Cost	Symbol	Cost
○	2	× or *	1
◇2	1/2	*2	1/4
◇3	2/3	*3	1/3
...
◇n	$\frac{n-1}{n}$	*n	$\frac{n-1}{2n}$

We then calculate the total “cost” of a signature by adding up the costs of each component. For example, a pattern with signature *333 has cost 2:

$$\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$$

Problem 47:

Calculate the costs of the following signatures:

- ◇3*3
- **
- ◇4*2:

Solution:

- ◇3*3: $\frac{2}{3} + 1 + \frac{1}{3} = 2$
- **: $1 + 1 = 2$
- ◇4*2: $\frac{3}{4} + 1 + \frac{1}{4} = 2$

Theorem 48:

The signatures of planar wallpaper patterns are exactly those with total cost 2.

We will not prove this theorem today, accept it without proof.

Problem 49:

Consider the 4 symmetries (translation, reflection, rotation, and glide reflection).

Which preserve orientation? Which reverse orientation?

Solution:

- Reflections and glide reflections reverse orientation (directions of spirals).
- Translation and rotation preserve orientation.

Problem 50:

Use the signature-cost theorem to find all the signatures consisting of only \circ or rotational symmetries.

Solution:

$\diamond 632, \diamond 442, \diamond 333, \diamond 2222, \circ$

Problem 51:

Find all the signatures consisting of only mirror symmetries.

Solution:

$*632, *442, *333, *2222, **$

Problem 52:

Find all the remaining signatures.

Each must be a mix of of mirror symmetries, rotational symmetries, or glide reflections.

Hint: They are all shown in the problems section.

Solution:

$\diamond 3*3, \diamond 4*2, \diamond 22\times, \diamond 22*, \times\times, \times*$