Lattices

Prepared by Mark on September 25, 2025

Instructor's Handout

This file contains solutions and notes.
Compile with the "nosolutions" flag before distributing.
Click [here] for the latest version of this handout.

Definition 1:
The *integer lattice* $\mathbb{Z}^n$ is the set of points with integer coordinates in n dimensions.
For example, $\mathbb{Z}^3$ is the set of points (a, b, c) where a, b, and c are integers.

Problem 2:
Draw $\mathbb{Z}^2$.

Definition 3:
We say a set of vectors $\{v_1, v_2, ..., v_k\}$ *generates* $\mathbb{Z}^n$ if every lattice point can be written as

$$a_1 v_1 + a_2 v_2 + ... + a_k v_k$$

for integer coefficients a_i.
Bonus: show that k must be at least n.

Problem 4:
Which of the following generate $\mathbb{Z}^2$?
- $\{(1, 2), (2, 1)\}$
- $\{(1, 0), (0, 2)\}$
- $\{(1, 1), (1, 0), (0, 1)\}$

Solution

Only the last.

Problem 5:
Find a set of two vectors other than $\{(0, 1), (1, 0)\}$ that generates $\mathbb{Z}^2$.

Problem 6:
Find a set of vectors that generates $\mathbb{Z}^n$.

Definition 7:
Say we have a generating set of a lattice.
The *fundamental region* of this set is the n-dimensional parallelogram spanned by its members.

Problem 8:
Draw two fundamental regions of $\mathbb{Z}^2$ using two different generating sets. Verify that their volumes are the same.

Part 1: Minkowski's Theorem

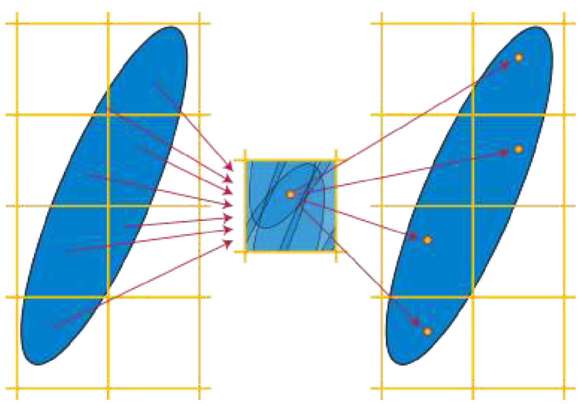
Theorem 9: Blichfeldt's Theorem
Let X be a finite connected region. If the volume of X is greater than 1, X must contain two distinct points that differ by an element of $\mathbb{Z}^n$. In other words, there exist distinct $x, y \in X$ so that $x - y \in \mathbb{Z}^n$.

Intuitively, this means that you can translate X to cover two lattice points at the same time.

Problem 10:
Draw a connected region in $\mathbb{R}^2$ with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint:* Area is two-dimensional volume.

Problem 11:
Draw a *disconnected* region in $\mathbb{R}^2$ with volume greater than 1 that contains no lattice points, and show that no two points in that region differ by an integer vector. In other words, show that Theorem 9 indeed requires a connected region.

Problem 12:
The following picture gives an idea for the proof of Blichfeldt's theorem in $\mathbb{Z}^2$.
Explain the picture and complete the proof.



Solution

The fundamental region of $\mathbb{Z}^2$ tiles the plane. Translate these tiles by lattice vectors to stack them on the fundamental region. Then since the union of the intersections of X with these tiles has area greater 1 and they are stacked on a region of area 1, there must be an overlap by a generalization of the pigeonhole principle (if there were no overlap then the sum of the areas would be less than or equal to 1). Take points x, y in the overlap. Then $x - y$ is a lattice point corresponding to the difference in translates, which were lattice points. Hence, $x - y \in \mathbb{Z}^2$.

Problem 13:
Let X be a region $\in \mathbb{R}^2$ of volume k. How many integral points must X contain after a translation?

Solution

$\lceil k \rceil$

Definition 14:
A region X is *convex* if the line segment connecting any two points in X lies entirely in X.

Problem 15:
Draw a convex region in two dimensions.
Then, draw a two-dimensional region that is *not* convex.

Definition 16:
We say a region X is *symmetric with respect to the origin* if for all points $x \in X$, $-x$ is also in X.
In the following problems, "*symmetric*" means "symmetric with respect to the origin."

Problem 17:
Draw a symmetric region.
Then, draw an asymmetric region.

Problem 18:
Show that a convex symmetric set always contains the origin.

Theorem 19: Minkowski's Theorem
Every convex set in $\mathbb{R}^n$ that is symmetric and has a volume
greater than 2^n contains an integral point that isn't zero.

Problem 20:
Draw a few sets that satisfy Theorem 19 in $\mathbb{R}^2$.
What is a simple class of regions that has the properties listed above?

Problem 21:
Let K be a region in $\mathbb{R}^2$ satisfying Theorem 19.
Let K' be this region scaled by $\frac{1}{2}$.

- How does the volume of K' compare to K?
- Show that the sum of any two points in K' lies in K *Hint:* Use convexity.
- Apply Blichfeldt's theorem to K' to prove Minkowski's theorem in $\mathbb{R}^2$.

Problem 22:
Let K be a region in $\mathbb{R}^n$ satisfying Theorem 19. Scale this region by $\frac{1}{2}$, called $K' = \frac{1}{2}K$.

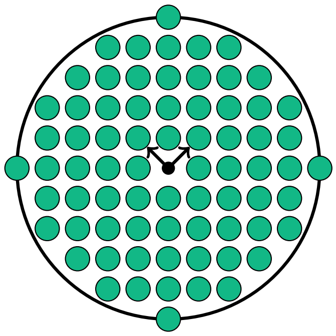
- How does the volume of K' compare to K?
- Show that the sum of any two points in K' lies in K
- Apply Blichfeldt's theorem to K' to prove Minkowski's theorem.

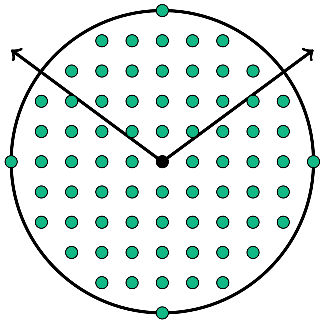
Solution

- The volume of K' is $\frac{1}{2^n}$ the volume of K.
- Take $x, y \in K'$. It follows that $2x, 2y \in K$. Since K is convex, we have that the midpoint of the line segment between $2x$ and $2y$ is in K, and so $\frac{2x+2y}{2} = x + y \in K$.
- Since the volume of K is greater than 2^n, we have the volume of K' is greater than one. Applying Blichfeldt's theorem, we can find two distinct points $x, y \in K'$ such that $x - y \in \mathbb{Z}^n$. Since K' is symmetric with respect to the origin, we have that $-y \in K'$. Therefore, $x + (-y) \in K$ by the previous part. $x \neq y, x - y \neq 0$, so we have found a nontrivial integer point in K.

Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius R. A tree of radius r has been planted at every integer point in the circle. If r is small, you will have a clear line of sight through the orchard. If r is large, there will be no clear line of sight through in any direction:





Problem 23:
Show that you will have at least one clear line of sight if $r < \frac{1}{\sqrt{R^2+1}}$.
Hint: Consider the line segment from $(0,0)$ to $(R,1)$. Calculate the distance from the closest integer points to the ray.

Solution

Consider the ray from the origin to the point $(R,1)$.
The two lattice points closest to this ray are $(1,0)$ and $(R-1,1)$. Say the distance from each of these points to the ray is δ.
Now, consider the triangle with vertices $(0,0)$, $(1,0)$, and $(R,1)$. The area of this triangle is $\frac{1}{2}$.
The area of this triangle is also equal to $\frac{1}{2}\delta\sqrt{R^2+1}$. By algebra, $\delta = \frac{1}{\sqrt{R^2+1}}$.
Therefore, if $r < \frac{1}{\sqrt{R^2+1}}$, we will have a clear line of sight given by this ray.

Problem 24:
Show that there is no line of sight through the orchard if $r > \frac{1}{R}$. You may want to use the following steps:

- Show that there is no line of sight if $r \geq 1$.
- Suppose $r < 1$ and $r > \frac{1}{R}$. Then, $R \geq 2$. Choose a potential line of sight passing through an arbitrary point P on the circle. Thicken this line of sight equally on both sides into a rectangle of width $2r$ tangent to P and $-P$. From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

Solution

Suppose $r < 1$ and let L be a potential line of sight. Consider the rectangle of width $2r$ tangent to P and $-P$. Then this is convex and symmetric with respect to the origin. Its area is $(2R)(2r) > 4\frac{R}{R} = 4$. By Minkowski, we have a nonzero integral point in this rectangle. Suppose first that the integer point is within the orchard. Then this means that there is a tree whose distance to the line is at most r. Therefore, this tree blocks the line of sight. Now notice that there is a part of this rectangle that sits outside the orchard. Can the integer point be in this region? This would mean its distance to the origin, D, would satisfy $D > R$. Now since this point is within a distance of r of our line L, we have that $D < \sqrt{R^2 + r^2} < \sqrt{R^2 + 1}$. So we have that $R < D < \sqrt{R^2 + 1}$. Then $R^2 < D^2 < R^2 + 1$, but D^2 is an integer so this is impossible.

Problem 25: Challenge
Prove that there exists a rational approximation of $\sqrt{3}$ within 10^{-3} with denominator at most 501. Come up with an upper bound for the smallest denominator of a ϵ-close rational approximation of any irrational number $\alpha > 0$. Your bound can have some dependence on α and should get smaller as α gets larger.
Hint: Use the orchard.

Solution

Take the line through the origin of slope $\sqrt{3}$. We would like an orchard for which $r = 10^{-3}$ gives no line of sight, since this will guarantee an integer point within a distance of 10^{-3}. Then by our previous problem, we can take $10^{-3} > \frac{1}{R}$, so take $R > 1000$. Now since this line intersects the boundary of the orchard at $(\frac{R}{2}, \frac{\sqrt{3}R}{2})$, we have that the x-coordinate is at most $\frac{R}{2} = 501$. Then we have that our lattice point (x, y) satisfies $\sqrt{3x} - 10^{-3} < y < \sqrt{3x} + 10^{-3}$, so $\sqrt{3} - 10^{-3} < \frac{x}{y} < \sqrt{3} + \frac{10^{-3}}{x}$. Therefore, $\frac{y}{x}$ is a rational approximation that is 10^{-3}-close to $\sqrt{3}$ and has denominator at most 501. Notice that we got closer than we need to.
Repeating this same process, our upper bound for the denominator of an ϵ-close approximation of α is $\frac{\cos(\text{atan}(\alpha))}{\epsilon} = \frac{1}{\sqrt{\alpha^2+1\epsilon}}$.