

# Warm-Up: Partition Products

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## Instructor's Handout

This handout contains solutions and notes.

Recompile without solutions before distributing.

### Problem 1:

Take any positive integer  $n$ .

Now, write it as sum of smaller positive integers:  $n = a_1 + a_2 + \dots + a_k$

Maximize the product  $a_1 \times a_2 \times \dots \times a_k$

### Solution:

#### Interesting Solution:

Of course, all  $a_i$  should be greater than 1.

Also, all  $a_i$  should be smaller than four, since  $x \leq x(x-2)$  if  $x \geq 4$ .

Thus, we're left with sequences that only contain 2 and 3.

Note that two twos are the same as one four, but we exclude fours for simplicity.

Finally, we see that  $3^2 > 2^3$ , so any three twos are better repackaged as two threes.

The best sequence  $a_i$  thus consists of a maximal number of threes followed by 0, 1, or 2 twos.

#### Calculus Solution:

First, solve this problem for equal, real  $a_i$ :

We know  $n = \prod(a_i)$ , thus  $\ln(n) = \sum(\ln(a_i))$ .

If all  $a_i$  are equal, we get  $\ln(n) = k \times \ln\left(\frac{n}{k}\right)$ .

Derive wrt  $k$  and set to zero to get  $\ln\left(\frac{n}{k}\right) = 1$

So  $k = \frac{n}{e}$  and  $\frac{n}{k} = e \approx 2.7$

If we try to approximate this with integers, we get the same solution as above.