Definable Sets

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Instructor's Handout

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Part 1: Logical Algebra

Definition 1:
Logical operators operate on the values {`true`, `false`},
just like algebraic operators operate on numbers.
In this handout, we'll use the following operators:

- $\neg$: not
- $\wedge$: and
- $\vee$: or
- $\rightarrow$: implies
- (): parenthesis.

The function of these is defined by *truth tables*:

and

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

or

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

implies

A	B	$A \rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

not

A	$\neg A$
T	F
F	T

$A \wedge B$ is `true` only if both A and B are `true`. $A \vee B$ is `true` if A or B (or both) are `true`.
$\neg A$ is the opposite of A, which is why it looks like a "negative" sign.

$A \rightarrow B$ is a bit harder to understand. Read aloud, this is "A implies B."
The only time $\rightarrow$ produces `false` is when `true` $\rightarrow$ `false`. This fact may seem counterintuitive, but will make more sense as we progress through this handout.
Hint: Think about it—if event α implies β, it is impossible for α to occur without β.
This is the only impossibility. All other variants are valid.

Problem 2:
Evaluate the following.

- $\neg$T
- F $\vee$ T
- T $\wedge$ T
- (T $\wedge$ F) $\vee$ T
- ($\neg$(F $\vee$ $\neg$T)) $\rightarrow$ $\neg$T
- (F $\rightarrow$ T) $\rightarrow$ ($\neg$F $\vee$ $\neg$T)

Solution

F T T T F T

Note for Instructors

We can also think of $[x \geq 0] \to b$ as follows: if x isn't the kind of object we care about, we evaluate true and check the next one. If x *is* the kind of object we care about and b is false, we have a counterexample to $[x \geq 0] \to b$, and thus $\mathtt{T} \to \mathtt{F}$ must be false.

Say we have the sentence $\forall x\ (a \to b)$.
For example, take $\varphi = \forall x\ ([x \geq 0] \to [\exists y\ y^2 = x])$.
φ holds whenever any positive x has a square root.

If $(\mathrm{F} \to *)$ returned false, statements like the above would be hard to write.
If x is negative, φ doesn't care whether or not it has a root. In this case, $\mathrm{F} \to *$ must be true to avoid making whole $\forall$ false.

Problem 3:
Evaluate the following.

- $A \to \mathtt{T}$ for any A
- $(\neg(A \to B)) \to A$ for any A, B
- $(A \to B) \to (\neg B \to \neg A)$ for any A, B

Note for Instructors

Note that the last formula is the contrapositive of $A \to B$.

Solution

All are true.

Problem 4:
Show that $\neg(A \to \neg B)$ is equivalent to $A \wedge B$.
That is, show that these expressions always evaluate to the same value given the same A and B.
Hint: Use a truth table

Problem 5:
Write an expression equivalent to $A \vee B$ using only $\neg$, $\to$, and ()?

Solution

$((\neg A) \to B)$

Note that both $\wedge$ and $\vee$ can be defined using the other logical symbols.
The only logical symbols we *need* are $\neg$, $\to$, and ().
We include $\wedge$ and $\vee$ to simplify our expressions.

Part 2: Structures

Definition 6:
A *universe* is a set of meaningless objects. Here are a few examples:

- $\{a, b, ..., z\}$
- $\{0, 1\}$
- $\mathbb{Z}$, $\mathbb{R}$, etc.

Definition 7:
A *structure* consists of a universe and a set of *symbols*.
A structure's symbols give meaning to the objects in its universe.

Symbols come in three types:

- *Constant symbols*, which let us specify specific elements of our universe.
  Examples: $0, 1, \frac{1}{2}, \pi$
- *Function symbols*, which let us navigate between elements of our universe.
  Examples: $+, \times, \sin x, \sqrt{x}$
  Note that symbols we usually call "operators" are functions under this definition.
  The only difference between $a + b$ and $+(a, b)$ is notation.
- *Relation symbols*, which let us compare elements of our universe.
  Examples: $<, >, \leq, \geq$

The equality check $=$ is *not* a relation symbol. It is included in every structure by default.
By definition, $a = b$ is true if and only if a and b are the same element of our universe.

Example 8:
The first structure we'll look at is the following:

$$\Big(\mathbb{Z} \mid \{0, 1, +, -, <\}\Big)$$

This is a structure over the universe $\mathbb{Z}$ that provides the following symbols:

- Constants: $\{0, 1\}$
- Functions: $\{+, -\}$
- Relations: $\{<\}$

If we look at our set of constant symbols, we see that the only integers we can directly refer to in this structure are 0 and 1. If we want any others, we must define them using the tools this structure offers.

To "define" an element of a set, we need to write a sentence that is only true for that element.
If we want to define 2 in the structure above, we could use the following sentence:

$$\text{"2 is the } x \text{ that satisfies } [1 + 1 = x]\text{."}$$

This is a valid definition because 2 is the *only* element of $\mathbb{Z}$ for which $[1 + 1 = x]$ evaluates to `true`.

Problem 9:
Define -1 in $\Big(\mathbb{Z} \mid \{0, 1, +, -, <\}\Big)$.

Solution

The sentences "x where $[x + 1 = 0]$" and "x where $[0 - 1 = x]$" both work.

Let us formalize what we found in the previous two problems.

Definition 10: Formulas
A *formula* in a structure S is a well-formed string of constants, functions, relations, and logical operators.

You already know what a "well-formed string" is: $1+1$ is fine, $\sqrt{+}$ is nonsense.
For the sake of time, I will not provide a formal definition — it isn't particularly interesting.

As a quick example, the formula $\psi \coloneqq [\neg(1=1)]$ is always false,
and $\varphi(x) \coloneqq [1+1=x]$ evaluates to `true` only when x is 2.

Definition 11: Free Variables
A formula can contain one or more *free variables.* These are denoted $\varphi(a, b, ...)$.
Formulas with free variables let us define "properties" that certain objects have.

For example, consider the two formulas from the previous definition, ψ and φ:

- $\psi \coloneqq [\neg(1=1)]$
  There are no free variables in this formula.
  In any structure, ψ is always either `true` or `false`.
- $\varphi(x) \coloneqq [1+1=x]$
  This formula has one free variable, labeled x.
  The value of $\varphi(x)$ depends on the x we're talking about:
  $\varphi(72)$ is false, and $\varphi(2)$ is true.

This "free variable" notation is very similar to the function notation we are used to:
The values of both $\varphi(x) \coloneqq [x > 0]$ and $f(x) = x+1$ depend on x.

Definition 12: Definable Elements
Let S be a structure over a universe U.
We say an element $x \in U$ is *definable in* S if we can write a formula $\varphi(x)$ that only x satisfies.

Problem 13:
Define 2 in the structure $\left(\mathbb{Z}^+ \mid \{4, \times\}\right)$.
Hint: $\mathbb{Z}^+ = \{1, 2, 3, ...\}$. Also, $2 \times 2 = 4$.

Solution

2 is the only element in $\mathbb{Z}^+$ that satisfies $\varphi(x) \coloneqq [x \times x = 4]$.

Problem 14:
Try to define 2 in the structure $\left(\mathbb{Z} \mid \{4, \times\}\right)$.
Why can't you do it?

Solution

We could try $\varphi(x) \coloneqq [x \times x = 4]$, but this is satisfied by both 2 and -2.
We have no way to distinguish between negative and positive numbers.
This problem is intentionally hand-wavy. We don't have the tools to write a proper proof.

Note for Instructors

Actually, it is. Bonus problem: how?
Do this after understanding quantifiers.

Problem 15:
Consider the structure $\left(\mathbb{R}_0^+ \mid \{1, 2, \div\}\right)$

- Define 2^2
- Define 2^n for all positive integers n
- Define 2^{-n} for all positive integers n
- What other numbers can we define in this structure?
  Hint: There is at least one more "class" of numbers we can define.

Solution

As far as I've seen, we can define any $2^{a/b}$ for $a, b \in \mathbb{Z}$.
For example, $\phi(x) \coloneqq [2 = x \div (1 \div x)]$ defines $\sqrt{2}$.

Part 3: Quantifiers

Recall the logical symbols we introduced earlier: $(), \wedge, \vee, \neg, \rightarrow$
We will now add two more: $\forall$ (for all) and $\exists$ (exists).

Definition 16:
$\forall$ and $\exists$ are *quantifiers*. They allow us to make statements about arbitrary symbols.
Quantifiers are aptly named: they tell us *how many* symbols satisfy a certain sentence.

Let's look at $\forall$ first. If $\varphi(x)$ is a formula,
the formula $\forall x\ \varphi(x)$ is true only if φ is true for all x in our universe.

For example, take the formula $\forall x\ (0 < x)$.
In English, this means "For any x, x is bigger than zero," or simply "Any x is positive."

$\exists$ is very similar: the formula $\exists x\ \varphi(x)$ is true if there is at least one x for which $\varphi(x)$ is true.
For example, $\exists\ (0 < x)$ means "there is a positive number in our set."

Problem 17:
Which of the following are true in $\mathbb{Z}$? Which are true in $\mathbb{R}_0^+$?
$\mathbb{R}_0^+$ is the set of positive real numbers and zero.

- $\forall x\ (x \geq 0)$
- $\neg(\exists x\ (x = 0))$
- $\forall x\ [\exists y\ (y \times y = x)]$
- $\forall xy\ \exists z\ (x < z < y)$ This is a compact way to write $\forall x\ (\forall y\ (\exists z\ (x < z < y)))$
- $\neg\exists x\ (\forall y\ (x < y))$

Solution

- "all x are positive" $\mathbb{R}_0^+$
- "zero doesn't exist" neither
- "square roots exist" $\mathbb{R}_0^+$
- "this set is dense" $\mathbb{R}_0^+$
- "there is no minimum" $\mathbb{Z}$

Problem 18:
Does the order of $\forall$ and $\exists$ in a formula matter?
What's the difference between $\exists x\ \forall y\ (x \leq y)$ and $\forall y\ \exists x\ (x \leq y)$?
Hint: Consider $\mathbb{R}^+_*$ the set of positive reals. Zero is not positive.
Which of the above formulas is true in $\mathbb{R}^+_*$ and which is false?

Solution

If $\exists x$ is inside $\forall y$, x depends on y. We may pick a different value of x for every y.
If $\exists x$ is outside, x is fixed *before* we check all y.

Problem 19:
Define 0 in $\Big(\mathbb{Z} \mid \{\times\}\Big)$

Solution

$\varphi(x) \coloneqq \big[\ \forall y\ x \times y = x\ \big]$

Problem 20:
Define 1 in $\Big(\mathbb{Z} \mid \{\times\}\Big)$

Solution

$\varphi(x) \coloneqq \big[\ \forall y\ x \times y = y\ \big]$

Problem 21:
Define -1 in $\Big(\mathbb{Z} \mid \{0, <\}\Big)$

Solution

$\varphi(x) \coloneqq \big[\ (x < 0) \wedge \neg\exists y\ (x < y < 0)\ \big]$

Problem 22:
Let $\varphi(x)$ be a formula.
Write a formula equivalent to $\forall x\ \varphi(x)$ using only logical symbols and $\exists$.

Solution

$\forall x\ \varphi(x)$ is true if and only if $\neg\exists x\ \neg\varphi(x)$ is true.

Part 4: Definable Sets

Armed with $(), \wedge, \vee, \neg, \rightarrow, \forall$, and $\exists$, we have the tools to define sets.

Definition 23: Set-Builder Notation
Say we have a sentence $\varphi(x)$.
The set of all elements that satisfy that sentence may be written as follows:

$$\{x \mid \varphi(x)\}$$

This is read "The set of x where φ is true" or "The set of x that satisfy φ."

For example, take the formula $\varphi(x) = \exists y\ (y + y = x)$.
The set of all even integers can then be written as

$$\{x \mid \exists y\ (y + y = x)\}$$

Definition 24: Definable Sets
Let S be a structure with a universe U.
We say a subset M of U is *definable* if we can write a formula
that is true for some x if and only if M contains x.

For example, consider the structure $(\mathbb{Z} \mid \{+\})$.
Only even numbers satisfy the formula $\varphi(x) \coloneqq [\exists y\ (y + y = x)]$,
so we can define "the set of even numbers" as $\{x \mid \exists y\ (y + y = x)\}$.
Remember—we can only use symbols that are available in our structure!

Problem 25:
The empty set is definable in any structure. How?

Solution

Always: $\{x \mid \neg(x = x)\}$

Problem 26:
Define $\{0, 1\}$ in $\left(\mathbb{Z}_0^+ \mid \{<\}\right)$ *Hint:* Define 0 and 1 as elements first, and remember that we can use logical symbols.

Solution

$\varphi_0(x) \coloneqq [\ \neg\exists y\ y < x\ ]$
$\varphi_1(x) \coloneqq [\ (0 < x)\ \wedge\ \neg\exists y\ (x < y < 0)\ ]$

Our final solution is $\{x \mid \varphi_0(x) \vee \varphi_1(x)\}$.
A finite set of definable elements is always definable.
An infinite set of definable elements might not be definable.

Problem 27:
Define the set of prime numbers in $\left(\mathbb{Z} \mid \{\times, \div, <\}\right)$.
Hint: A prime number is an integer that is positive and is only divisible by 1 and itself.

Solution

$\psi(x) \coloneqq [\ \exists y\ (0 < y < x)\ ]$ "x is positive and isn't 0 or 1"
$\varphi(x) \coloneqq [\ (x < 0) \wedge \neg\exists ab\ (\psi(a) \wedge \psi(b) \wedge a \times b = x)]$

Problem 28:
Define $\mathbb{R}_0^+$ in $\Big(\mathbb{R} \mid \{\times\}\Big)$

Solution

$\varphi(x) \coloneqq \big[\exists y \; y \times y = x\big]$

Problem 29:
Let $\triangle$ be a relational symbol. $a \triangle b$ is only true if a divides b.
Define the set of prime numbers in $\Big(\mathbb{Z}^+ \mid \{\triangle\}\Big)$

Solution

$\varphi(x) \coloneqq \big[\; \neg\exists abc \; \big((a \triangle x) \wedge (b \triangle x) \wedge (c \triangle x) \wedge \neg(a = b) \wedge \neg(a = c) \wedge \neg(b = c)\big) \;\big]$

Theorem 30: Lagrange's Four Square Theorem
Every natural number may be written as a sum of four integer squares.

Problem 31:
Define $\mathbb{Z}_0^+$ in $\Big(\mathbb{Z} \mid \{\times, +\}\Big)$

Solution

$\varphi(x) \coloneqq \big[\; \exists abcd \; (a^2 + b^2 + c^2 + d^2 = x) \;\big]$, where $a^2 \coloneqq a \times a$.

Problem 32:
Define $<$ in $\Big(\mathbb{Z} \mid \{\times, +\}\Big)$
Hint: We can't formally define a relation yet. Don't worry about that for now.
You can repharase this question as "given $x, y \in \mathbb{Z}$, write a formula $\varphi(x, y)$ that is only true if $x < y$"

Solution

Let $\psi(x)$ be the formula from the previous problem.

$\varphi(x, y) \coloneqq \big[\; \neg(x = y) \wedge \exists d \; \big(\psi(d) \wedge (x + d = y)\big) \;\big]$

Problem 33:
Consider the structure $S = (\mathbb{R} \mid \{0, \diamond\})$
The relation $a \diamond b$ holds if $|a - b| = 1$

Part 1:
Define $\{-1, 1\}$ in S.

Solution

$\varphi(x) \coloneqq \big[0 \diamond x\big]$

Part 2:
Define $\{-2, 2\}$ in S.

Solution

$\varphi(x) \coloneqq \big[\ \forall a\ (0 \diamond x \to a \diamond x) \wedge \neg(x = 0)\ \big]$

Problem 34:
Let $\mathcal{P}$ be the set of all subsets of $\mathbb{Z}_0^+$. This is called the *power set* of $\mathbb{Z}_0^+$.
Let S be the structure $(\mathcal{P} \mid \{\subseteq\})$

Part 1:
Show that the empty set is definable in S.
Hint: Defining $\{\}$ with $\{x \mid \neg x = x\}$ is **not** what we need here.
We need $\varnothing \in \mathcal{P}$, the "empty set" element in the power set of $\mathbb{Z}_0^+$.

Solution

$\varphi(x) \coloneqq \big[\ \forall y\ x \subseteq y\ \big]$
Note that we can use the same property to define 0 in $(\mathbb{Z} \mid \{\leq\})$

Part 2:
Let $x \fallingdotseq y$ be a relation on $\mathcal{P}$. $x \fallingdotseq y$ holds if $x \cap y \neq \{\}$.
Show that $\fallingdotseq$ is definable in S.

Solution

Let $\psi(x)$ be the formula from the previous problem.

$\varphi(x, y) \coloneqq \big[\ \exists x\ (a \subseteq x) \wedge (a \subseteq y) \wedge \neg\psi(a)\ \big]$

Part 3:
Let f be the function on $\mathcal{P}$ defined by $f(x) = \mathbb{Z}_0^+ - x$. This is called the *complement* of x.
Show that f is definable in S.
Hint: You can define a function by writing a formula $\varphi(x, y)$ that is only true when $y = f(x)$.

Part 5: Equivalence

Notation:
Let S be a structure and φ a formula.
If φ is true in S, we write $S \models \varphi$.
This is read "S satisfies φ"

Definition 35:
Let S and T be structures.
We say S and T are *equivalent* (and write $S \equiv T$) if for any formula φ, $S \models \varphi \iff T \models \varphi$.
If S and T are not equivalent, we write $S \not\equiv T$.

Problem 36:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{R} \mid \{+, 0\}\Big)$

Problem 37:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{N} \mid \{+, 0\}\Big)$

Problem 38:
Show that $\Big(\mathbb{R} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{N} \mid \{+, 0\}\Big)$

Problem 39:
Show that $\Big(\mathbb{R} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{Z}^2 \mid \{+, 0\}\Big)$

Problem 40:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{Z}^2 \mid \{+, 0\}\Big)$

Solution

All of the above are easy, but the last one can take a while.
The trick is to notice that $\mathbb{Z}$ has two equivalence classes mod 2, while $\mathbb{Z}^2$ has four.