

# Combinatorics

Prepared by Mark on January 25, 2025

---

## Part 1: Getting started

An **ordered** arrangement of objects is called a *permutation*.

An **unordered** selection of objects is called a *combination*.<sup>1</sup>

All the following problems involve permutations.

### Problem 1:

How many different ways are there to rearrange the letters ABCDE?

### Problem 2:

How many different ways are there to arrange the letters ABCDEFG...XYZ?

You don't need to fully evaluate your answer, it is a *very* big number.

*Hint:* Look at ?? again, and try to create a general strategy.

---

<sup>1</sup>A “combination lock” cares about the order of its digits, so its name is inaccurate. Such an object is actually a *permutation* lock!

**Definition 1:**

The *factorial* of a positive integer  $x$  is  $x \times (x - 1) \times \dots \times 1$ . We denote this  $x!$ .  
For example,  $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$ .

**Problem 3:**

Compute  $\frac{10!}{8!}$

**Problem 4:**

Convince yourself that  $(n + 1)! = n! \times (n + 1)$ , and use this fact to show that  $0! = 1$ .

**Problem 5:**

How many ways are there to choose three student council officers from a class of 20 students?

How many ways are there to choose a president, a vice-president, and a treasurer from the same class?

*Hint:* Your answers should be different. In which case does order matter?

**Problem 6:**

Say you have 4 red balls and 3 green balls. How many different ways can you arrange them on the table in front of you?

**Problem 7:**

How many *unique* anagrams can we create from the word CRESCENDO?

**Problem 8:**

Given the letters ABCDE, how many different three-letter words can we make without repeating letters?

## Part 2: Permutations

It would be convenient to have a general tool for counting permutations. Let us try to create one. (Remember, permutations are *ordered* arrangements of objects.)

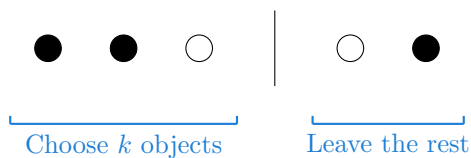
First, let's create a function  ${}_nP_k$ , which tells us how many  $k$ -object permutations we can choose from a group of  $n$  objects.

### Problem 9:

What is  ${}_5P_3$ ?

*Hint:* See ??

“Choosing  $k$  items from  $n$ ” is a lot like splitting our  $n$  objects into two groups: those we choose, and those we don't.



If we rearrange these, we get different permutations. How can we count them?

### Problem 10:

Using the above diagram, create a formula for  ${}_nP_k$ .

*Hint:* We're counting *permutations*, so the order of items in the first group matters.

## Part 3: Combinations

Now, let's count *combinations*.

Here, we only care about *which* items we choose—not the order in which we choose them. We'll make a function  ${}_nC_k$  (“n choose k”), which will tell us how many different ways we can choose  $k$  items from a set of  $n$ .

### Problem 11:

Find an expression for  ${}_nC_k$  by modifying your definition of  ${}_nP_k$ .

Usually,  ${}_nC_k$  is written as  $\binom{n}{k}$ . This is also called the *binomial coefficient*.

### Problem 12:

Say you have a few coins on the table in front of you:

- 8 identical 1-kopek<sup>2</sup> coins
- 3 identical 2-kopek coins
- 6 identical 5-kopek coins
- 4 identical 10-kopek coins

How many distinct ways are there to arrange these coins in a row?

### Problem 13:

Now, derive the *multinomial coefficient*  $\binom{n}{k_1, k_2, \dots, k_m}$ .

The multinomial coefficient tells us how many distinct ways there to arrange  $n$  objects of  $m$  classes, where each class  $i$  contains  $k_i$  identical objects.

*Hint:* In ??,  $n = 21$  and  $(k_1, k_2, k_3, k_4) = (8, 3, 6, 4)$ .

So, the solution to ?? should be given by the multinomial coefficient  $\binom{21}{8, 3, 6, 4}$ .

---

<sup>2</sup>Russian currency. Comparable to a penny, since 100 kopeks make a ruble.

## Part 4: Applications

**Problem 14:**

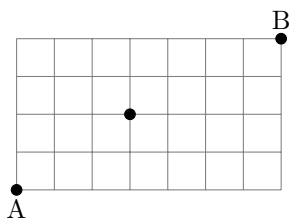
How many ways can a class of 27 people be seated in 30 seats?

**Problem 15:**

The following is the map of a city. Each line is a one-way road, you can only drive up or right.

How many different paths can you take from A to B?

How many of them go through the center point?



**Problem 16:**

How many ways can you put 19 identical balls into 6 bins, leaving no bin empty?

**Problem 17:**

Given an exam with 4 problems, how many ways are there to assign positive point values to each problem so that the exam contains a total of 100 points?

**Problem 18:**

How many ways can we split the number 2016 into a sum of positive integers?

Consider  $2016 + 1$  and  $1 + 2016$  distinct sums. Order matters.

**Problem 19:**

A staircase must be built up a wall. It will start 4.5 meters away from the wall, which is 1.5 meters tall. The height of each step is exactly 30 centimeters. The width of each step must be an integer multiple of 50 centimeters. In how many ways can the staircase be constructed?