Random Walks and Resistance

Prepared by Mark on January 25, 2025
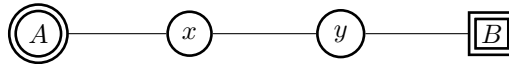Based on a handout by Aaron Anderson

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Instructor's Handout

This file contains solutions and notes.
Compile with the "nosolutions" flag before distributing.
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Part 1: Random Walks

Consider the graph below. A particle sits on some node n. Every second, this particle moves left or right with equal probability. Once it reaches node A or B, it stops.
We would like to compute the probability of our particle stopping at node A.

In other words, we want a function $P : \text{Nodes} \to [0, 1]$ that maps each node of the graph to the probability that our particle stops at A.

$$A \quad\text{—}\quad x \quad\text{—}\quad y \quad\text{—}\quad B$$

Problem 1:
What are $P(A)$ and $P(B)$ in the graph above?
Note that these values hold for all graphs.

> **Solution**
>
> $P(A) = 1$ and $P(B) = 0$

Problem 2:
Find an expression for $P(x)$ in terms of $P(y)$ and $P(A)$.
Find an expression for $P(y)$ in terms of $P(x)$ and $P(B)$.

> **Solution**
>
> $P(x) = \frac{P(A)+P(y)}{2}$
>
> $P(y) = \frac{P(B)+P(x)}{2}$

Problem 3:
Use the previous problems to find $P(x)$ and $P(y)$.

> **Solution**
>
> $P(x) = {}^2\!/_3$
>
> $P(y) = {}^1\!/_3$

Problem 4:

Say we have a graph G and a particle on node x with neighbors $v_1, v_2, ..., v_n$.
Assume that our particle is equally likely to travel to each neighbor.
Find $P(x)$ in terms of $P(v_1), P(v_2), ..., P(v_n)$.

Solution

We have
$$P(x) = \frac{P(v_1) + P(v_2) + ... + P(v_n)}{n}$$

Problem 5:

In general, how do we find $P(n)$ for any node n?

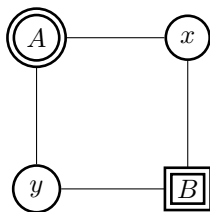### Solution

If we write an equation for each node other than A and B, we have a system of $|N| - 2$ linear equations in $|N| - 2$ variables.

We still need to show that this system is nonsingular, but that's outside the scope of this handout. This could be offered as a bonus problem.

Problem 6:

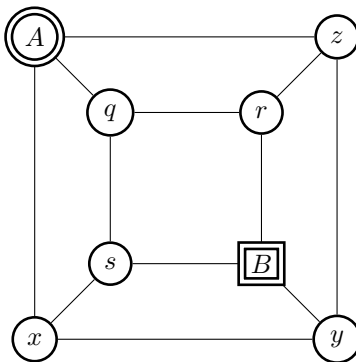Find $P(n)$ for all nodes in the graph below.

Problem 7:

Find $P(n)$ for all nodes in the graph below.

Note that this is the graph of a cube with A and B on opposing vertices.

Definition 8:

Let us now take a look at weighted graphs. The problem remains the same: we want to compute the probability that our particle stops at node A, but our graphs will now feature weighted edges. The probability of our particle taking a certain edge is proportional to that edge's weight.

For example, if our particle is on node y of the graph below, it has a $\frac{3}{8}$ probability of moving to x and a $\frac{1}{8}$ probability of moving to z.
Note that $3 + 3 + 1 + 1 = 8$.

Problem 9:

Say a particle on node x has neighbors $v_1, v_2, ..., v_n$ with weights $w_1, w_2, ..., w_n$.
The edge (x, v_1) has weight w_1. Find $P(x)$ in terms of $P(v_1), P(v_2), ..., P(v_n)$.

Solution

$$P(x) = \frac{w_1 P(v_1) + w_2 P(v_2) + ... + w_n P(v_n)}{w_1 + w_2 + ... + w_n}$$
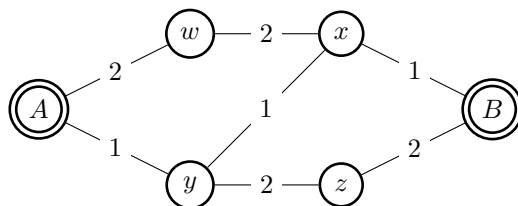
Problem 10:
Consider the following graph. Find $P(x)$, $P(y)$, and $P(z)$.

Problem 11:
Consider the following graph.
What expressions can you find for $P(w)$, $P(x)$, $P(y)$, and $P(z)$?

Solve this system of equations.
Hint: Use symmetry. $P(w) = 1 - P(z)$ and $P(x) = 1 - P(y)$. Why?

Part 2: Circuits

An *electrical circuit* is a graph with a few extra properties, called *current*, *voltage*, and *resistance*. In the definitions below, let X be the set of nodes in a circuit.

- **Voltage** is a function $V : X \to \mathbb{R}$ that assigns a number to each node of our graph.
  In any circuit, we pick a "ground" node, and define the voltage[1] there as 0.
  We also select a "source" node, and define its voltage as 1.

  Intuitively, you could say we're connecting the ends of a 1-volt battery to our source and ground nodes.

- **Current** is a function $I : X^2 \to \mathbb{R}$ that assigns a number to each *oriented edge* in our graph.
  An "oriented edge" is just an ordered pair of nodes (n_1, n_2).

  Current through an edge (a, b) is a measure of the flow of charge from a to b.
  Naturally, $I(a, b) = -I(b, a)$.

- **Resistance** is a function $R : X^2 \to \mathbb{R}_0^+$ that represents a certain edge's resistance to the flow of current through it.
  Resistance is a property of each *link* between nodes, so order doesn't matter: $R(a, b) = R(b, a)$.

It is often convenient to compare electrical circuits to systems of pipes. Say we have a pipe from point A to point B. The size of this pipe represents resistance (smaller pipe $\implies$ more resistance), the pressure between A and B is voltage, and the speed water flows through it is to current.

Definition 12: Ohm's law
With this "pipe" analogy in mind, you may expect that voltage, current, and resistance are related: if we make our pipe bigger (and change no other parameters), we'd expect to see more current. This is indeed the case! Any circuit obeys *Ohm's law*, stated below:

$$V(a, b) = I(a, b) \times R(a, b)$$

This handout uses two notations for voltage: two-variable $V(a, b)$ and one-variable $V(a)$.
The first represents the voltage between points a and b, better reflecting reality (see the footnote below). The second measures the voltage between a and ground, and is more convenient to use in equations. **Try to use the single-variable notation in your equations.** Convince yourself that $V(a, b) = V(a) - V(b)$.

Definition 13: Kirchoff's law
The second axiom of electrical circuits is also fairly simple. *Kirchoff's law* states that the sum of all currents connected to a given edge is zero. You can think of this as "conservation of mass": nodes in our circuit do not create or destroy electrons, they simply pass them around to other nodes. Formally, we can state this as follows:

Let x be a node in our circuit and N_x the set of its neighbors. We than have

$$\sum_{b \in N_x} I(x, b) = 0$$

which must hold at every node **except the source and ground vertices.**
Hint: Keep this exception in mind, it is used in a few problems later on.

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[1]In the real world, voltage is always measured *between two points* on a circuit. Voltage is defined as the *difference* in electrical charge between two points. Hence, voltage is a function of two nodes.

Note that this is different than current and resistance, which aren't functions of two arbitrary nodes — rather, they are functions of *edges* (i.e, two adjacent nodes).

Consider the circuit below. **This the graph from Problem 1**, turned into a circuit by:
- Replacing all edges with 1Ω resistors
- Attaching a 1 volt battery between A and B

Note that the battery between A and B isn't really an edge. It exists only to create a potential difference between the two nodes.

Problem 14:
From the circuit diagram above, we immediately know that $V(A) = 1$ and $V(B) = 0$.
What equations related to the currents out of x and y does Kirchoff's law give us?
Hint: Current into x = current out of x

Problem 15:
Use Ohm's law to turn the equations from Problem 14 into equations about voltage and resistance. Find an expression for $V(x)$ and $V(y)$ in terms of other voltages, then solve the resulting system of equations. Does your result look familiar?

Solution

$$V(x) = \frac{V(A) - V(y)}{2}$$
$$V(y) = \frac{V(x) - V(B)}{2}$$

Part 3: The Equivalence

In the last problem, we found that the equations for $V(x)$ were the same as the equations for $P(x)$ on the same graph. It turns out that this is true in general: problems about voltage in circuits directly correspond to problems about probability in graphs. We'll spend the next section proving this fact.

Definition 16:
For the following problems, *conductance* will be more convenient than resistance.
The definition of conductance is quite simple:

$$C(a,b) = \frac{1}{R(a,b)}$$

Aside: Resistance is usually measured in Ohms, denoted Ω.
A few good-natured physicists came up with the "mho" (denoted $\mho$) as a unit of conductance, which is equivalent to an inverse Ohm. Unfortunately, NIST discourages the use of Mhos in favor of the equivalent (and less amusing) "Siemens."

Problem 17:
Let x be a node in a graph.
Let N_x be the set of x's neighbors, $w(x,y)$ the weight of the edge between nodes x and y, and W_x the sum of the weights of all edges connected to x.
We saw earlier that the probability function P satisfies the following sum:

$$P(x) = \sum_{b \in N_x} \left(P(b) \times \frac{w(x,b)}{W_x} \right)$$

This was never explicitly stated, but is noted in Definition 8.

Use Ohm's and Kirchoff's laws to show that the voltage function V satisfies a similar sum:

$$V(x) = \sum_{b \in N_x} \left(V(b) \times \frac{C(x,b)}{C_x} \right)$$

where $C(x,b)$ is the conductance of edge (x,b) and C_x is the sum of the conductances of all edges connected to x.

Solution

First, we know that
$$\sum_{b \in N_x} I(x,b) = 0$$

for all nodes x. Now, substitute $I(x,b) = \frac{V(x)-V(b)}{R(x,y)}$ and pull out $V(x)$ terms to get

$$V(x) \sum_{b \in N_x} \frac{1}{R(x,b)} - \sum_{b \in N_x} \frac{V(b)}{R(x,b)} = 0$$

Rearranging and replacing $R(x,b)^{-1}$ with $C(x,b)$ and $\sum C(x,b)$ with C_x gives us

$$V(x) = \sum_{b \in N_x} V(b) \frac{C(x,b)}{C_x}$$

Thus, if $w(a, b) = C(a, b)$, P and V satisfy the same system of linear equations. To finish proving that $P = V$, we now need to show that there can only be one solution to this system. We will do this in the next two problems.

Problem 18:
Let q be a solution to the following equations, where $x \neq a, b$.

$$q(x) = \sum_{b \in N_x} \left(q(b) \times \frac{w(x, b)}{W_x} \right)$$

Show that the maximum and minimum of q are $q(a)$ and $q(b)$ (not necessarily in this order).

> **Solution**
>
> The domain of q is finite, so a maximum and minimum must exist.
>
> Since $q(x)$ is a weighted average of all $q(b)$, $b \in N_x$, there exist $y, z \in N_x$ satisfying $q(y) \leq q(x) \leq q(z)$. Therefore, none of these can be an extreme point.
>
> A and B are the only vertices for which this may not be true, so they must be the minimum and maximum.

Problem 19:
Let p and q be functions that solve our linear system
and satisfy $p(A) = q(A) = 1$ and $p(B) = q(B) = 0$.

Show that the function $p - q$ satisfies the equations in Problem 18,
and that $p(x) - q(x) = 0$ for every x. Note that $p(x) - q(x) = 0 \ \forall x \implies p = q$

> **Solution**
>
> The equations in Problem 18 for p and q directly imply that
>
> $$[p - q](x) = \sum_{b \in N_x} \left([p - q](b) \times \frac{w(x, b)}{W_x} \right)$$
>
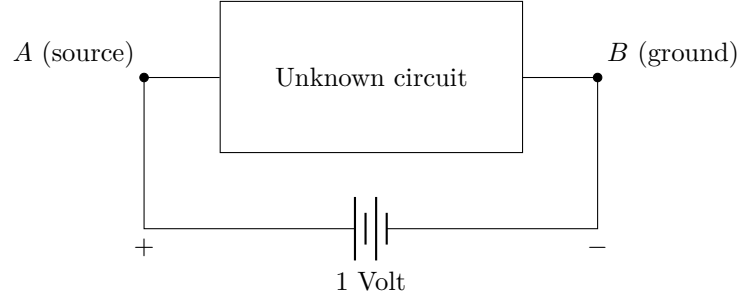> Which are the equations from Problem 18 for $(p - q)$.
>
> Hence, the minimum and maximum values of $p - q$ are $[p - q](a) = 1 - 1 = 0$ and $[p - q](b) = 1 - 1 = 0$.
>
> Therefore $p(x) - q(x) = 0$ for all x, so $p(x) = q(x)$.

Part 4: Effective Resistance

As we have seen, calculating the properties of a circuit by creating an equation for each vertex is a fairly time-consuming ordeal. Fortunately, there is a better strategy we can use.

Consider a graph (or a circuit) with source and ground vertices. All parts of the circuit that aren't these two vertices are hidden inside a box, as shown below:

What do we know about this box? If this was a physical system, we'd expect that the current flowing out of A is equal to the current flowing into B.

Problem 20:
Using Kirchoff's law, show that the following equality holds.
Remember that we assumed Kirchoff's law holds only at nodes other than A and B.
Note: As before, N_x is the set of neighbors of x.

$$\sum_{b \in N_A} I(A, b) = \sum_{b \in N_B} I(b, B)$$

Solution

Add Kirchoff's law for all vertices $x \neq A$ to get

$$\sum_{\forall x} \left(\sum_{b \in N_x} I(x, b) \right) = 0$$

This sum counts both $I(x, y)$ and $I(x, y)$ for all edges x, y, except $I(x, y)$ when x is A or B. Since $I(a, b) + I(b, a) = 0$, these cancel out, leaving us with
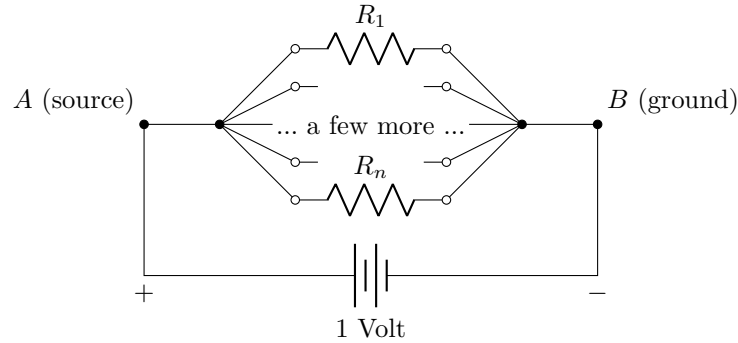
$$\sum_{b \in N_A} I(A, b) + \sum_{b \in N_B} I(B, b) = 0$$

Rearrange and use the fact that $I(a, b) = -I(b, a)$ to get the final equation.

If we call this current $I_A = \sum_{b \in N_A} I(A, b)$, we can pretend that the box contains only one resistor, carrying I_A units of current. Using this information and Ohm's law, we can calculate the *effective resistance* of the box.

Problem 21: Resistors in parallel

Using Ohm's law and Kirchoff's law, calculate the effective resistance R_{eff} of the circuit below.

Solution

Let I_i be the current across resistor R_i, from left to right.
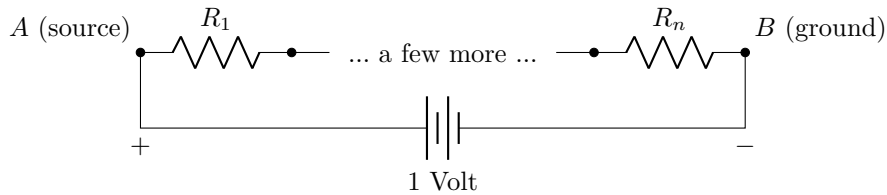By Ohm's law, $I_i = \frac{V}{R_i}$ (Note that $V = 1$ in this problem).

The source current is then $I_A = \sum_{i=1}^{n} = \left(V\right)\left(\frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}\right)$. Applying Ohm's law again, we find that

$$R_{\text{eff}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}}$$

Problem 22: Resistors in series

Using Ohm's law and Kirchoff's law, calculate the effective resistance R_{total} of the circuit below.

Solution

This solution uses the same notation as the solution for Problem 21.

By Kirchoff's law, all I_i are equal in this circuit. Let's say $I = I_i$.
Let V_i denote the voltage at the node to the left of R_i.
By Ohm's law, $V_i - V_{i+1} = IR_i$.

The sum of this over all i telescopes, and we get $V(A) - V(B) = I(R_1 + R_2 + ... + R_n)$.
Dividing, we find that

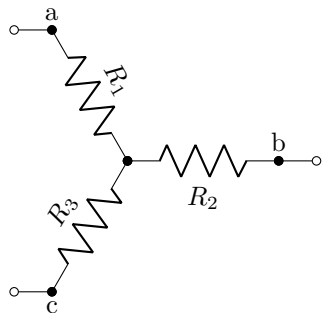$$R_{\text{eff}} = R_1 + R_2 + ... + R_n$$

We can now use effective resistance to simplify complicated circuits. Whenever we see the above constructions (resistors in parallel or in series) in a graph, we can replace them with a single resistor of appropriate value.
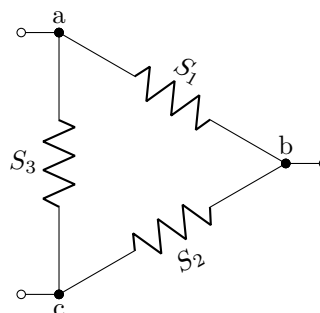
Problem 23:

Consider the following circuits. Show that the triangle has the same effective resistance as the star if

- $x = R_1 R_2 + R_1 R_3 + R_2 R_3$
- $S_1 = {}^x/_{R_3}$
- $S_2 = {}^x/_{R_1}$
- $S_3 = {}^x/_{R_2}$

The star: **The triangle:**

Problem 24:
Suppose we construct a circuit by connecting the 2^n vertices of an n-dimensional cube with 1Ω resistors. If we place A and B at opposing vertices, what is the effective resistance of this circuit?
Bonus: As $n \to \infty$, what happens to R_{eff}?
Note: Leave your answer as a sum.

Think of the vertices of the n-dimensional cube as n-bit binary strings, with A at `000...0` and B at `111...1`. We can divide our cube into $n+1$ layers based on how many ones are in each node's binary string, with the kth layer having k ones. By symmetry, all the nodes in each layer have the same voltage. This means we can think of the layers as connected in series, with the resistors inside each layer connected in parallel.

There are $\binom{n}{k}$ nodes in the kth layer. Each node in this layer has k ones, so there are $n - k$ ways to flip a zero to get to the $(k + 1)^{\text{th}}$ layer. In total, there are $\binom{n}{k}(n - k)$ parallel connections from the kth layer to the $(k + 1)^{\text{th}}$ layer, creating an effective resistance of

$$\frac{1}{\binom{n}{k}(n - k)}$$

The total effective resistance is therefore

$$\sum_{k=0}^{n-1} \frac{1}{\binom{n}{k}(n - k)}$$

To calculate the limit as $n \to \infty$, note that

$$\binom{n}{k}(n - k) = \frac{n!}{(n - k - 1)! \times k!} = n\binom{n - 1}{k}$$

So, the sum is

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}$$

Note that for $n \geq 4$, $\binom{n}{k} \geq \binom{n}{2}$ for $2 \leq k \leq n - 2$, so

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \leq 2 + \frac{2}{n} + \frac{n - 3}{\binom{n}{2}}$$

which approaches 2 as $n \to \infty$. So, $R_{\text{eff}} \to 0$ as $n \to \infty$.