Origami

Prepared by everyone on January 25, 2025

Instructor's Handout

This file contains solutions and notes.
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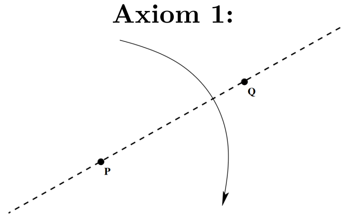
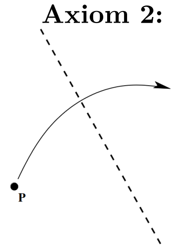
Part 1: Axioms of Origami

Axiom 1:



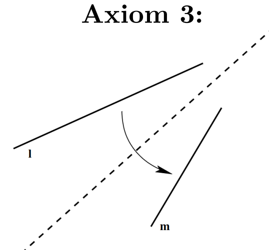
Given two points, we can fold a line between them.

Axiom 2:



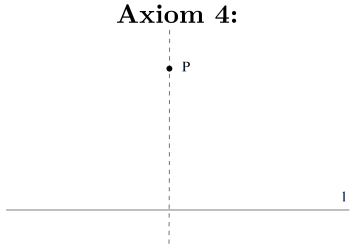
Given two points, we can make a fold that places one atop the other.

Axiom 3:



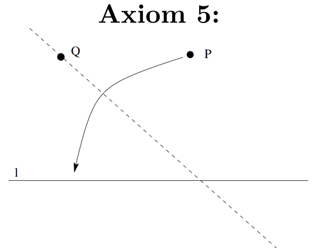
Given two lines, we can make a fold that places one atop the other

Axiom 4:



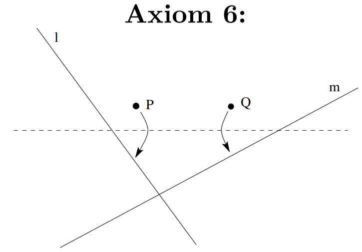
Given a point and a line, we can make a fold through the point and perpendicular to the line.

Axiom 5:



Given two points and a line, we can make a fold through one point that places the second on the line.

Axiom 6:

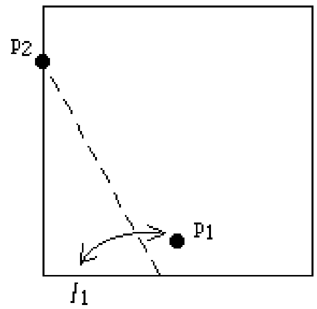


Given two points and two lines, we can make a fold that places each point on a line.

Problem 1:

Proposed by Nikita

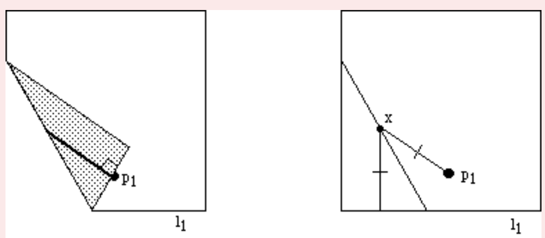
a: Take a piece of paper. Let the bottom edge be l_1 and take p_1 to be a point in the middle and close to l_1. Then choose p_2 to be anywhere on the left or right edge of the square and perform Axiom 5. Then choose a different p_2. Repeat this 8 or 9 times keeping the same p_1 and choosing different p_2's. What do you see?



b: Then, take another piece of paper. Draw two random intersecting lines l_1 and l_2 and points p_1 and p_2 about an inch close to their intersection. Perform a Beloch fold for them.

Solution

A: The repeated use of Axiom 5 in this exercise will result in the appearance of a parabola on the paper. Really!



To see why, imagine making one of the folds in this exercise. Before you unfold the flap, take a heavy black pen and draw a line from the point p_1 to the folded edge, making it perpendicular to the "folded-up" segment of l_1 (as in the left picture above). If our pen is heavy enough, this line will bleed through the paper and mark the underneath side as well, so when we unfold this flap we'll see two lines (as in the right picture above). Note that these two lines have the same length and one is perpendicular to the original line l_1. This shows that exactly one point on the crease line we just made is equidistant to the point p_1 and the line l_1. In other words, the crease line is tangent to the parabola with focus p_1 and directrix l_1. This should seem amazing - origami actually allows us to do simple calculus! Just one fold computes a tangent line of a parabola.

But there's a more important thing to observe here. Parabolas are given by second-degree equations. Thus Axiom 5 finds a point for us on some second-degree equation. In other words, Axiom 5 solves second-degree equations for us! It may seem strange to think of an origami fold as solving an equation, but mathematically this is exactly what is going on.

B: If you look closely, Axiom 6 is just like Axiom 5 but times two: In Axiom 5 p_1 is the focus and l_1 is the directrix of a parabola. In Axiom 6 we have this again, but also p2 is the focus and l_2 is the directrix of another parabola! Thus Axiom 6 solves the following problem: Given two parabolas drawn in the plane, find a line that is tangent to both of them.

Exercise: Show that doing this is equivalent to solving a 3rd-degree equation.

Problem 2:

Proposed by James

- **a:** Given a circle, its center, and a point p on the circle, use origami to construct a tangent line to the circle that passes through p.
- **b:** Given a circle, its center, and a point p on the circle, use origami to construct an equilateral triangle inscribed in the circle that passes through p.
- **c:** Given a triangle, use origami to construct the center of the circle inscribed in it and its tangent points.

Solution

A: Take the fold through the center of the circle and the point p, and take the perpendicular fold passing through p.

B: Fold p to the center of the circle to produce two points on the circle, do the same for the two points. The equilateral triangle doesn't pass through p. But by repeating the process for some of the vertices produced on the circle, one can produce an equilateral triangle that passes through p.

C: Fold one side to another side to produce the three angle bisectors. They intersect at a point, which is the incenter. Then use Axiom 4 for each of the three sides.

Problem 3:

Proposed by Nikita

Use origami to find the other three notable points in the given triangle: circumcenter, centroid and orthocenter.

Solution

Use Axiom 2 for pairs of vertices to get the circumcenter.
The creases from the perpendicular bisectors leave marks at the centers of sides which we use (in Axiom 1) to build the medians and the centroid. Use Axiom 4 for vertices and opposite sides to get the orthocenter.

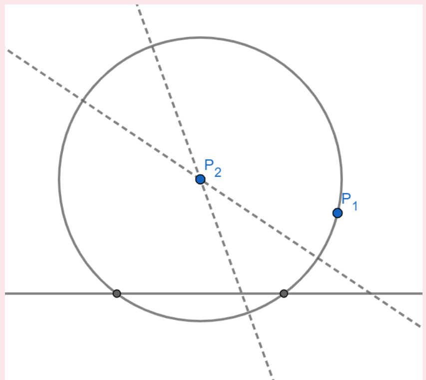
Problem 4:
Proposed by Nikita

a: Emulate Axiom 5 with a compass and straightedge.

b: In your emulation, probably, there is a choice of which of the two intersections of a circle and a line to take. Does it mean that there are two ways to perform the fold?

Solution

There are two ways to do it:



Problem 5:
Proposed by Nikita

Prove that $\sqrt[3]{2} \neq \frac{a}{b}$ for any $a, b \in \mathbb{N}$.

Solution

Suppose that $\sqrt[3]{2} = \frac{a}{b}$. Take the smallest such pair (a, b). Then $2b^3 = a^3$. So a is even. Let $a = 2c$. Then $2b^3 = 8c^3 \implies b^3 = 4c^3$. So b is also even, which contradicts the minimality assumption.

Problem 6:

Proposed by Nikita

a: Construct a regular hexagon using a ruler and compass.

b: Cut the triangle with angles 72°, 72°, 36° into 2 isosceles triangles.

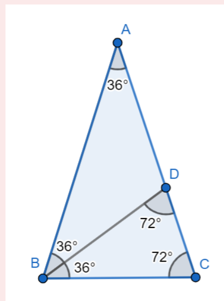
c: Using triangle similarity, prove that the ratio of the sides in this triangle is equal to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

d: Find a way to construct a regular pentagon using only a ruler and compass.

Solution

A: Obvious

B:



C: Let $|BC| = 1$. Then $|AC| = x$, the ratio we seek. From equilateral triangles we get $|AD| = |BD| = |BC| = 1$. From similar triangles we get $|CD| = \frac{|CD|}{|BD|} = \frac{1}{x}$. So $|AC| = |AD| + |DC|$ means $x = 1 + \frac{1}{x}$ or $x^2 - x - 1 = 0$. Solving for x we get $x = \frac{1 \pm \sqrt{5}}{2}$. We take the positive root.

D: Draw any segment BC and call its length the unit length. Then construct the right triangle with sides 1 and 2. This will give a segment of length $\sqrt{5}$. Using it we construct a segment of length $\varphi = \frac{1+\sqrt{5}}{2}$ and can find the vertex A of a regular pentagon since it should be distance φ apart from B and C. It's easy to find two other vertices.

Problem 7:

Proposed by Nikita

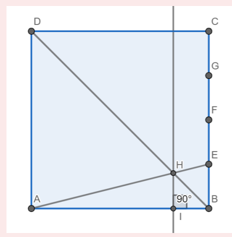
a: Use origami to divide a given segment into 3 equal parts.

b: Use origami to divide a given segment into n equal parts.

Solution

A: Build a square on this segment and then use the construction from the origami paper (or see the next part).

B: The same construction works inductively.



If in unit square $ABCD$ the length $|BE| = \frac{1}{n}$, then for $H = AE \cap BD$ the distance to BC is $\frac{1}{n+1}$, since $\triangle ADH \sim \triangle EBH$.

Problem 8:

Proposed by ?

a: In the lecture, you saw that Axioms 1–5 are all able to be simulated by compass and straightedge constructions. Is the following claim a correct *deduction* from the above? (In other words, does simulating Axioms 1–5 *prove* the claim?)
Claim: "In all cases, origami constructions are at least as powerful as compass and straightedge constructions."

b: Is the claim true? Argue *both* sides with yourself (or with a classmate).

c: (Hard) Prove the sense in which the claim is true. (Hint: recall from the lecture that all constructible lengths with straightedge and compass are rational, or of the form $a + b\sqrt{c}$ with a, b, c rational, or of the form $d + e\sqrt{f}$ with d, e, f of the form $a + b\sqrt{c}$ with a, b, c rational, etc.)

Solution

A: No, this argument leaves open the possibility that something can be constructed with a compass and straightedge, but cannot be with origami.

B: Argument for false: With a compass, one can draw circles, and origami can never draw curves that are not straight (in a finite number of steps.)
Argument for true: Although the pictures look different, it is possible (and actually true) that the constructible *lengths* with straightedge and compass form a strict subset of lengths with origami.

C: It suffices to construct such numbers with origami, i.e. that rationals, square roots, addition, and multiplication are all constructible.

These actually require a bit of critical thinking. See Lemma 4.3.3 (page 38/pdf 47) of ORIGAMI-CONSTRUCTIBLE NUMBERS by HWA YOUNG LEE (`https://getd.libs.uga.edu/pdfs/lee_hwa-young_201712_ma.pdf`) for the details.

Problem 9:

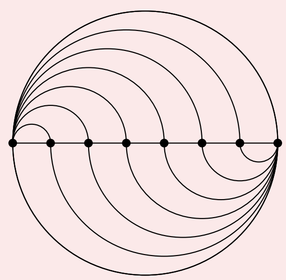
Proposed by Mark

Do each of the following with a compass and ruler.
Do not use folds.

a: Divide a circle into five parts of equal area.
b: Divide a circle into seven parts of equal area.
c: Divide a circle into n parts of equal area.

Solution

a: Trivial
b: Hard, since we can't make a 7-gon using a compass and a ruler. Use **c**.
c: Draw a diameter AB. Split that diameter into n equal segments. In the top half of the original circle, draw a half-circle from each point to A. In the bottom, do the same for B.

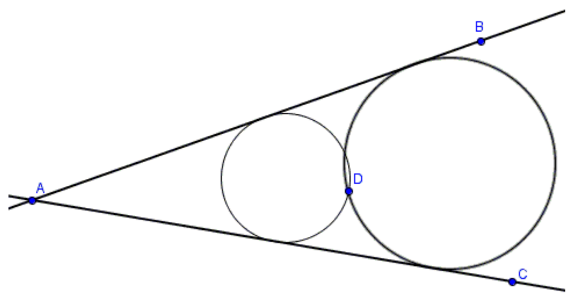


Problem 10:

Proposed by Sunny

Using a compass and ruler, find two circles tangent to a point D and lines AB and AC. (Problem of Appolonius, PLL case)

Hint: All circles tangent to AB and AC are homothetic with centre at A. What does this mean? Also, the angle bisector may help.



Solution

See diagram and instructions below.

- **A:** Draw angle bisector AE and line AD.
- **B:** Draw circle tangent to AB and AC centered at E.
- **C:** Draw EG and EH from E to line AD.
- **D:** Draw DI and DK parallel to EH and EG respectively (they are necessarily parallel by homothety). The desired circles are centered at I and K.