Lattices

Prepared by Mark on June 9, 2026

Definition 1:
The *integer lattice* $\mathbb{Z}^n$ is the set of points with integer coordinates in n dimensions.
For example, $\mathbb{Z}^3$ is the set of points (a, b, c) where a, b, and c are integers.

Problem 2:
Draw $\mathbb{Z}^2$.

Definition 3:
We say a set of vectors $\{v_1, v_2, ..., v_k\}$ *generates* $\mathbb{Z}^n$ if every lattice point can be written as

$$a_1v_1 + a_2v_2 + ... + a_kv_k$$

for integer coefficients a_i.
Bonus: show that k must be at least n.

Problem 4:
Which of the following generate $\mathbb{Z}^2$?
- $\{(1, 2), (2, 1)\}$
- $\{(1, 0), (0, 2)\}$
- $\{(1, 1), (1, 0), (0, 1)\}$

Problem 5:
Find a set of two vectors other than $\{(0, 1), (1, 0)\}$ that generates $\mathbb{Z}^2$.

Problem 6:
Find a set of vectors that generates $\mathbb{Z}^n$.

Definition 7:
Say we have a generating set of a lattice.
The *fundamental region* of this set is the n-dimensional parallelogram spanned by its members.

Problem 8:
Draw two fundamental regions of $\mathbb{Z}^2$ using two different generating sets. Verify that their volumes are the same.

Part 1: Minkowski's Theorem

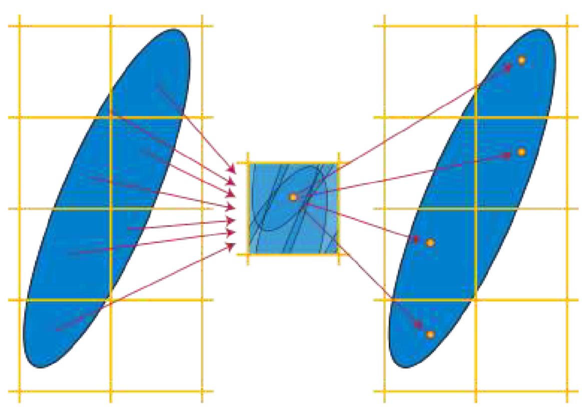
Theorem 9: Blichfeldt's Theorem
Let X be a finite connected region. If the volume of X is greater than 1, X must contain two distinct points that differ by an element of $\mathbb{Z}^n$. In other words, there exist distinct $x, y \in X$ so that $x - y \in \mathbb{Z}^n$.

Intuitively, this means that you can translate X to cover two lattice points at the same time.

Problem 10:
Draw a connected region in $\mathbb{R}^2$ with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint:* Area is two-dimensional volume.

Problem 11:
Draw a *disconnected* region in $\mathbb{R}^2$ with volume greater than 1 that contains no lattice points, and show that no two points in that region differ by an integer vector. In other words, show that **??** indeed requires a connected region.

Problem 12:
The following picture gives an idea for the proof of Blichfeldt's theorem in $\mathbb{Z}^2$.
Explain the picture and complete the proof.



Problem 13:
Let X be a region $\in \mathbb{R}^2$ of volume k. How many integral points must X contain after a translation?

Definition 14:
A region X is *convex* if the line segment connecting any two points in X lies entirely in X.

Problem 15:
Draw a convex region in two dimensions.
Then, draw a two-dimensional region that is *not* convex.

Definition 16:
We say a region X is *symmetric with respect to the origin* if for all points $x \in X$, $-x$ is also in X.
In the following problems, "*symmetric*" means "symmetric with respect to the origin."

Problem 17:
Draw a symmetric region.
Then, draw an asymmetric region.

Problem 18:
Show that a convex symmetric set always contains the origin.

Theorem 19: Minkowski's Theorem
Every convex set in $\mathbb{R}^n$ that is symmetric and has a volume greater than 2^n contains an integral point that isn't zero.

Problem 20:
Draw a few sets that satisfy **??** in $\mathbb{R}^2$.
What is a simple class of regions that has the properties listed above?

Problem 21:
Let K be a region in $\mathbb{R}^2$ satisfying **??**.
Let K' be this region scaled by $\frac{1}{2}$.

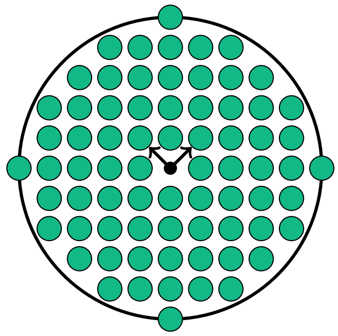
- How does the volume of K' compare to K?
- Show that the sum of any two points in K' lies in K *Hint:* Use convexity.
- Apply Blichfeldt's theorem to K' to prove Minkowski's theorem in $\mathbb{R}^2$.

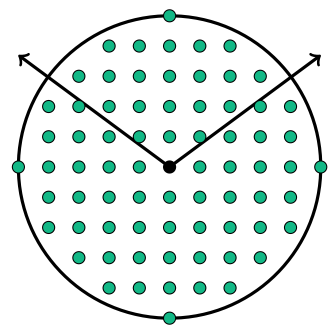
Problem 22:
Let K be a region in $\mathbb{R}^n$ satisfying **??**. Scale this region by $\frac{1}{2}$, called $K' = \frac{1}{2}K$.

- How does the volume of K' compare to K?
- Show that the sum of any two points in K' lies in K
- Apply Blichfeldt's theorem to K' to prove Minkowski's theorem.

Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius R. A tree of radius r has been planted at every integer point in the circle. If r is small, you will have a clear line of sight through the orchard. If r is large, there will be no clear line of sight through in any direction:





Problem 23:
Show that you will have at least one clear line of sight if $r < \frac{1}{\sqrt{R^2+1}}$.
Hint: Consider the line segment from $(0,0)$ to $(R,1)$. Calculate the distance from the closest integer points to the ray.

Problem 24:
Show that there is no line of sight through the orchard if $r > \frac{1}{R}$. You may want to use the following steps:

- Show that there is no line of sight if $r \geq 1$.
- Suppose $r < 1$ and $r > \frac{1}{R}$. Then, $R \geq 2$. Choose a potential line of sight passing through an arbitrary point P on the circle. Thicken this line of sight equally on both sides into a rectangle of width $2r$ tangent to P and $-P$. From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

Problem 25: Challenge
Prove that there exists a rational approximation of $\sqrt{3}$ within 10^{-3} with denominator at most 501. Come up with an upper bound for the smallest denominator of a ϵ-close rational approximation of any irrational number $\alpha > 0$. Your bound can have some dependence on α and should get smaller as α gets larger.
Hint: Use the orchard.