Wallpaper Symmetry

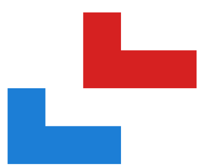
Prepared by Mark on June 5, 2026

Section 1: Wallpaper Symmetries

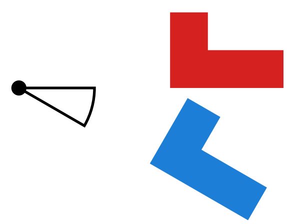
Definition 1:
A *Euclidean isometry* is a transformation of the plane that preserves distances.
Intuitively, an isometry moves objects on the plane without deforming them.
There are four classes of Euclidean isometries:

- translations
- reflections
- rotations
- glide reflections

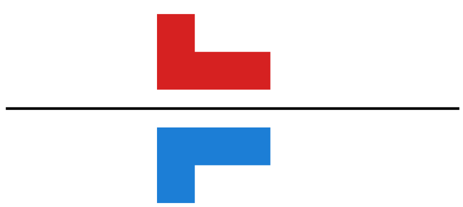
We can prove there are no others, but this is beyond the scope of this handout.
A simple example of each isometry is shown below:



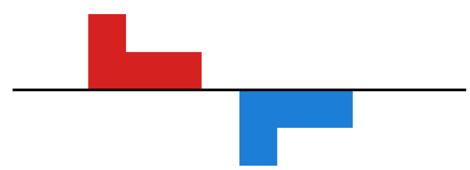
Translation



Rotation



Reflection



Glide reflection

Definition 2:
A *wallpaper* is a two-dimensional pattern that...

- has translational symmetry in at least two non-parallel directions (and therefore fills the plane)
  "Translational symmetry" means that we can slide the entire wallpaper in some direction, eventually mapping the pattern to itself.
- has a countable number of reflection, rotation, or glide symmetries.

Problem 3:
Is a plain square grid a valid wallpaper?

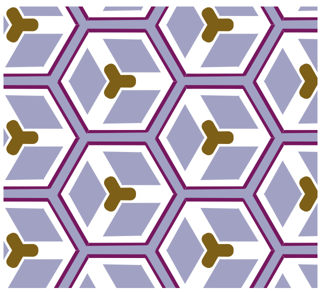
Problem 4:
Is the empty plane a valid wallpaper?

Section 2: Mirror Symmetry

Definition 5:
A *reflection* is a transformation of the plane obtained by reflecting all points about a line.
If this reflection maps the wallpaper to itself, we have a *mirror symmetry*.
If n such mirror lines intersect at a point, they form a *mirror node of order n*.
Mirror nodes with order 1 do not exist (i.e, $n \geq 2$). A line does not intersect itself!

Two mirror nodes on a wallpaper are identical if we can map one to the other with a translation and a rotation while preserving the pattern on that wallpaper.

Problem 6:
Find all three distinct mirror nodes in the following pattern.
What is the order of each node?
Hint: You may notice rotational symmetry in this pattern.
Don't worry about that yet, we'll discuss it later.



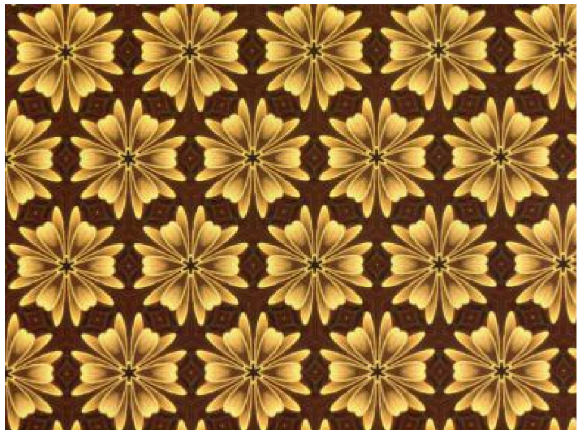
Definition 7:
Orbifold notation gives us a way to describe the symmetries of a wallpaper.
It defines a *signature* that fully describes all the symmetries of a given pattern.
We will introduce orbifold notation one symmetry at a time.

Definition 8:
In orbifold notation, mirror nodes are denoted by a $*$ followed by a list of integer.
Every integer n following a $*$ denotes a mirror node of order n.

The order of these integers doesn't matter. $*234$ and $*423$ are the same signature.
However, we usually denote n-fold symmetries in descending order (that is, like $*432$).
If we have many nodes of the same order, integers may be repeated.

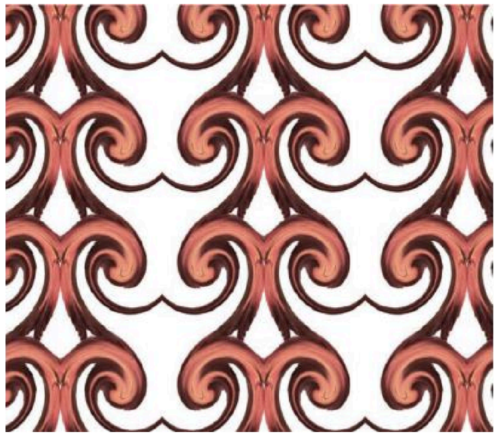
Problem 9:
What is the signature of the wallpaper in Problem 6?
Hint: Again, ignore rotational symmetry for now.

Problem 10:
Find the signature of the following pattern.



Problem 11:
Draw a wallpaper pattern with signature $*2222$

Remark 12:
In an exceptional case, we have two parallel mirror lines.
Consider the following pattern:



The signature of this pattern is $**$

Problem 13:
Draw another wallpaper pattern with signature $**$.

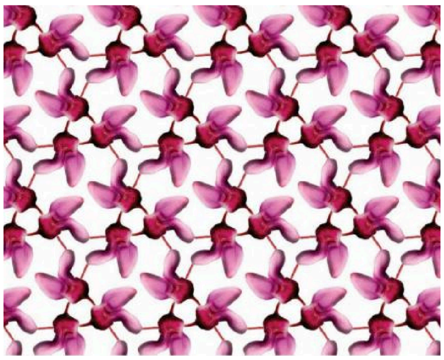
Section 3: Rotational Symmetry

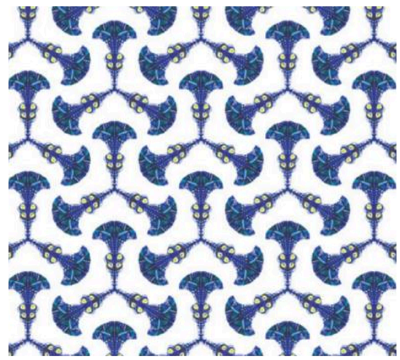
Definition 14:
A wallpaper may also have n-fold rotational symmetry about a point.

This means there are no more than n rotations around that point that map the wallpaper to itself.

As before, two points of rotational symmetry are identical if we can perform a translation and rotation that maps one to the other without changing the wallpaper.

Definition 15:
In orbifold notation, rotation is specified similarly to reflection, but uses the prefix $\diamond$.
For example:
- `◇333` denotes a pattern with three distinct centers of rotation of order 3.
- `◇4*2` denotes a pattern with one rotation center of order 4 and one mirror node of order 2.





Problem 16:
Find the three rotation centers in the left wallpaper.
What are their orders?

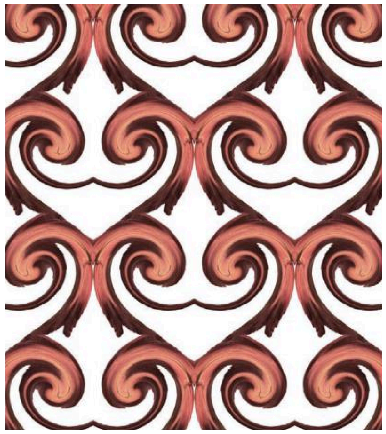
Problem 17:
Find the signature of the pattern on the right.

Remark 18:
You may have noticed that we could have an ambiguous classification, since two reflections are equivalent to a translation and a rotation. We thus make the following distinction: *rotational symmetry that can be explained by reflection is not rotational symmetry.*

In other words, when classifying a pattern…
- we first find all mirror symmetries,
- then all rotational symmetries that are not accounted for by reflection.

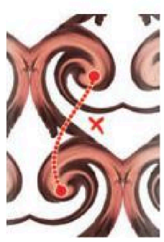
Section 4: Glide Reflections

Definition 19:
Another type of symmetry is the *glide reflection*, denoted $\times$.
A glide reflection is the result of a translation along a line followed by reflection about that line.
For example, consider the following pattern:



Problem 20:
Convince yourself that all mirror lines in this pattern are *not* distinct. / In other words, this pattern has only one mirror symmetry.

Problem 21:
Use the following picture to find the glide reflection in the above pattern.



Remark 22:
The signature of this wallpaper is $*\times$.

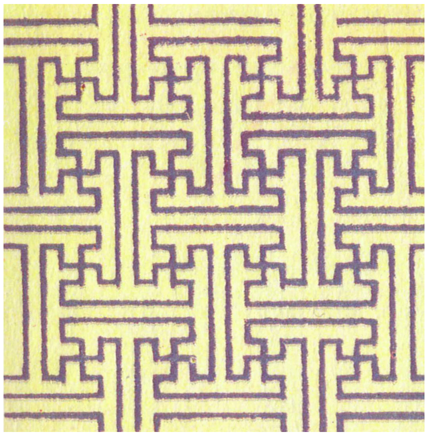
Definition 23:
If none of the above symmetries appear in a pattern, then we only have simple translational symmetry. We denote this with the signature ∘.

Remark 24:
In summary, to find the signature of a pattern:

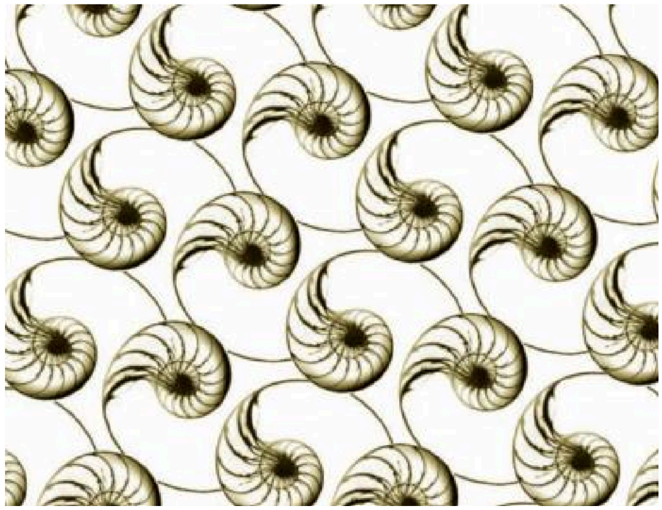
- find the mirror lines (∗) and the distinct intersections;
- then find the rotation centers (⋄) not explained by reflection;
- then find all glide reflections (×) that do not cross a mirror line.
- If we have none of the above, our pattern must be ∘.

Problem 25:
Find the signature of the following pattern:

Problem 26:
Find the signature of the following pattern:



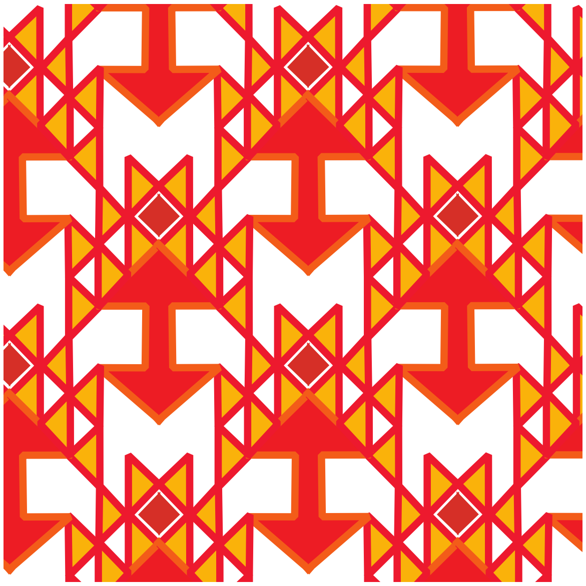
Problem 27:
Find two glide reflections in the following pattern.
(and thus show that its signature is $\times\times$.)

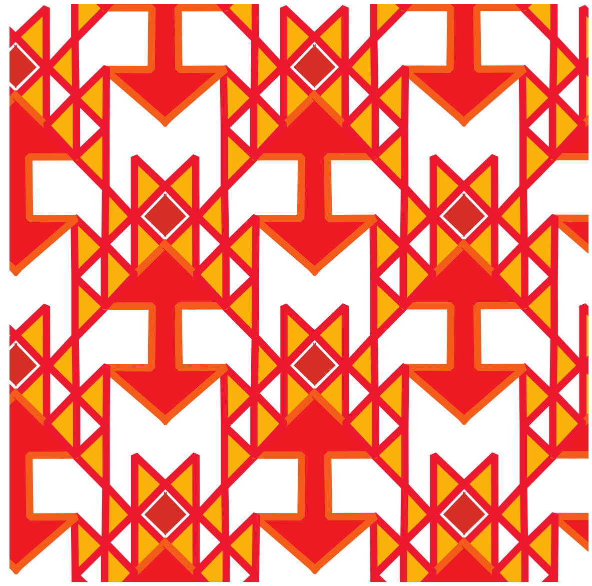


Section 5: A few problems

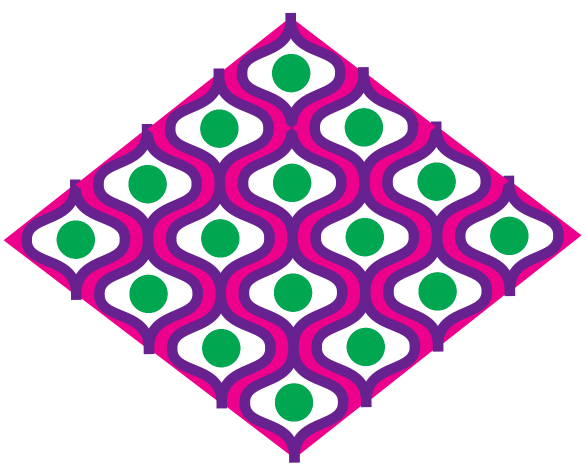
Find the signatures of the following patterns. Mark all mirror nodes, rotation centers, and glide reflections. Each pattern is provided twice for convenience.

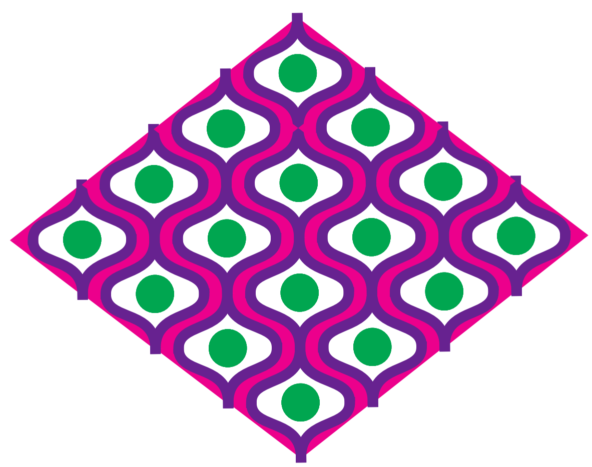
Problem 28:



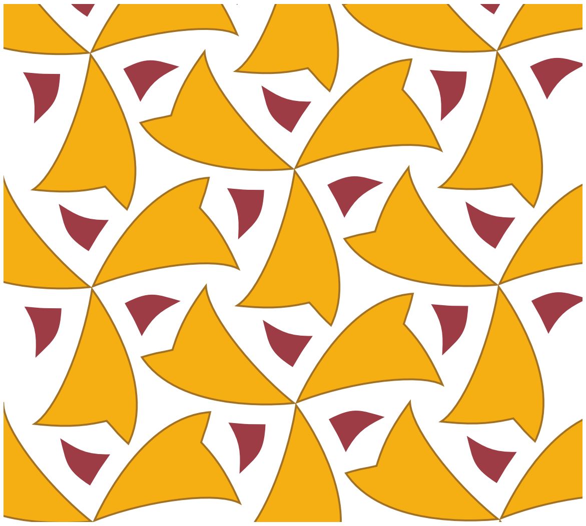


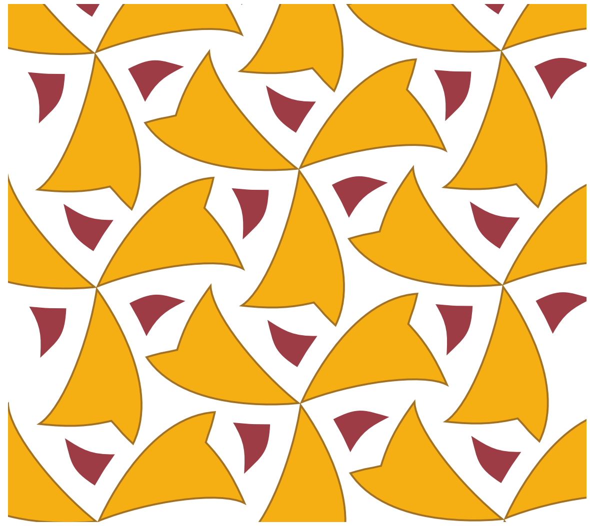
Problem 29:



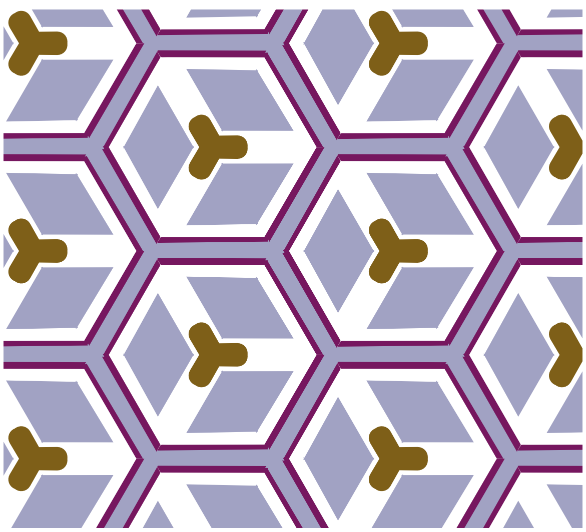


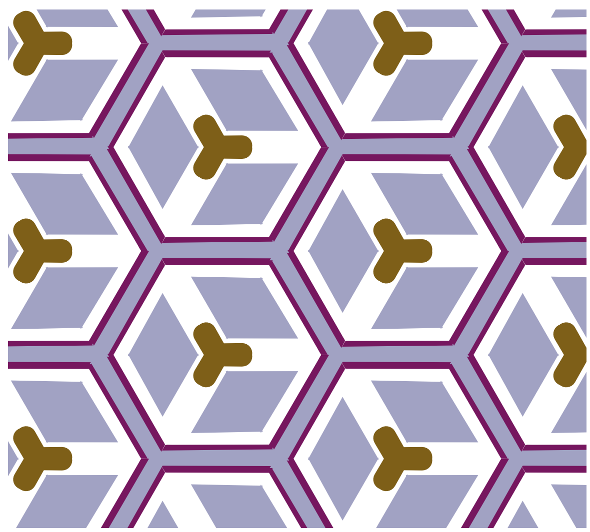
Problem 30:



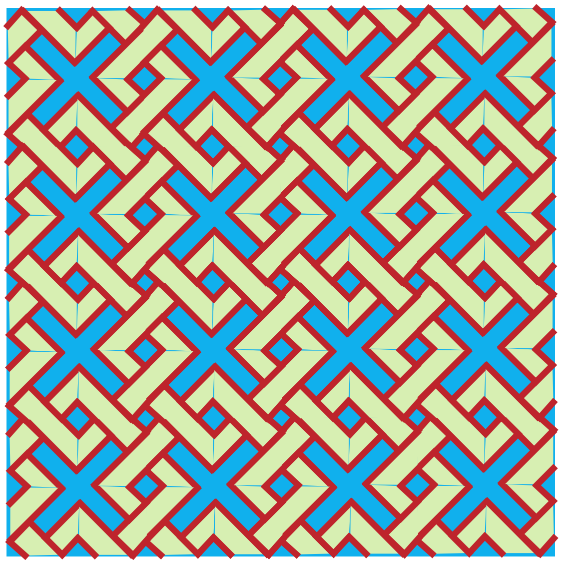


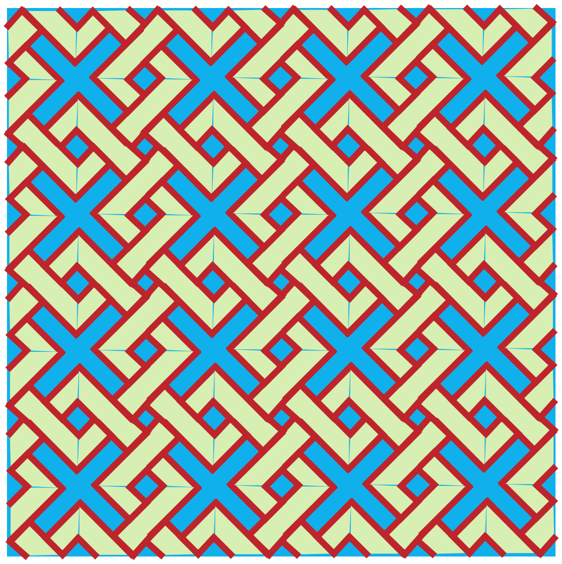
Problem 31:



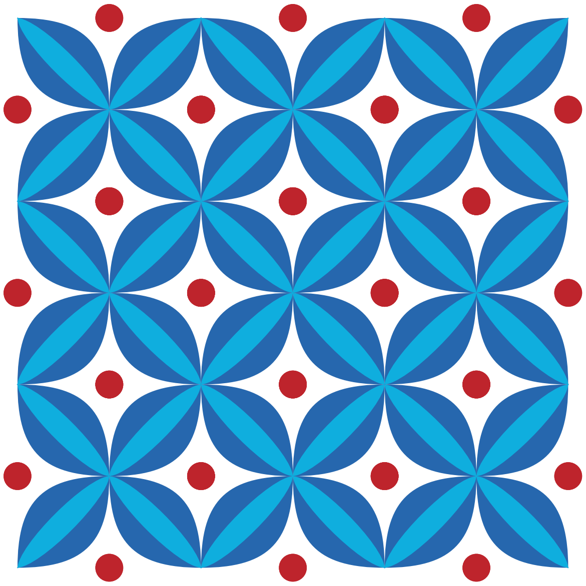


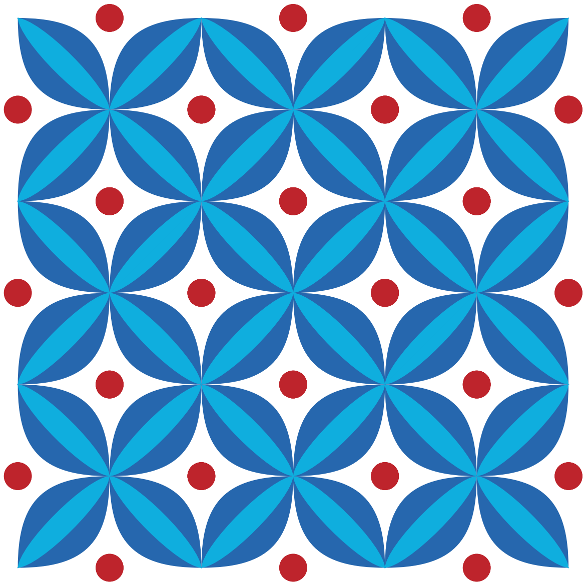
Problem 32:



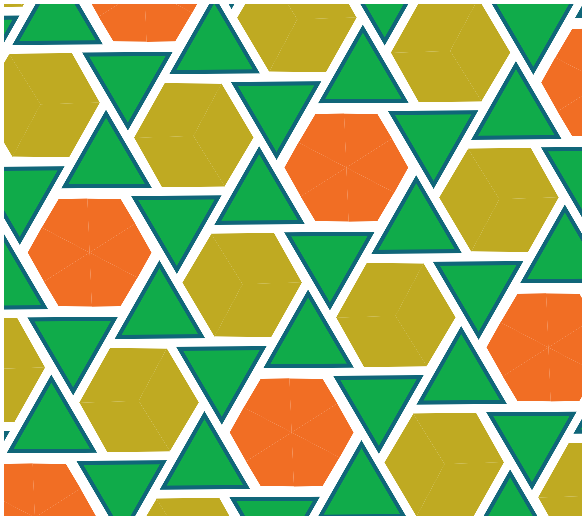


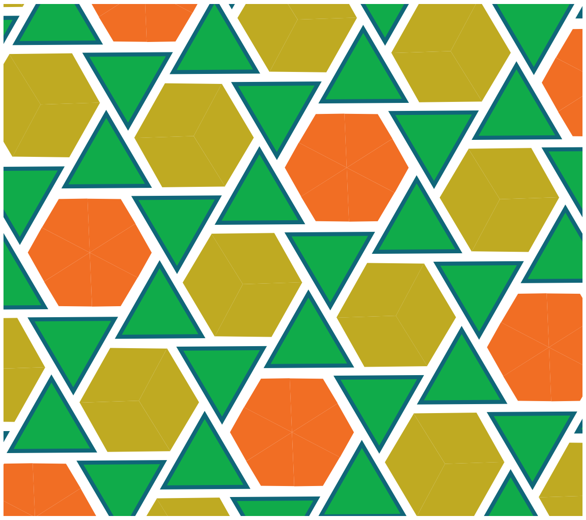
Problem 33:



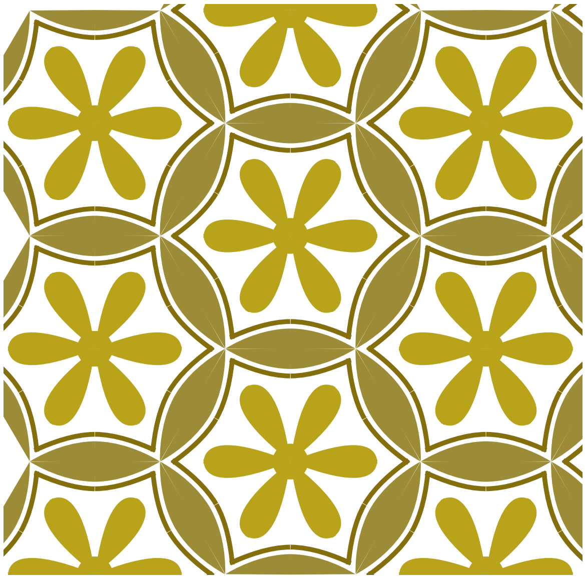


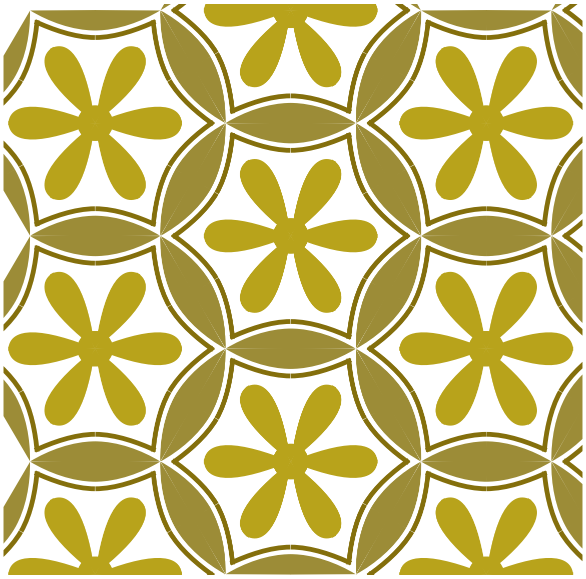
Problem 34:



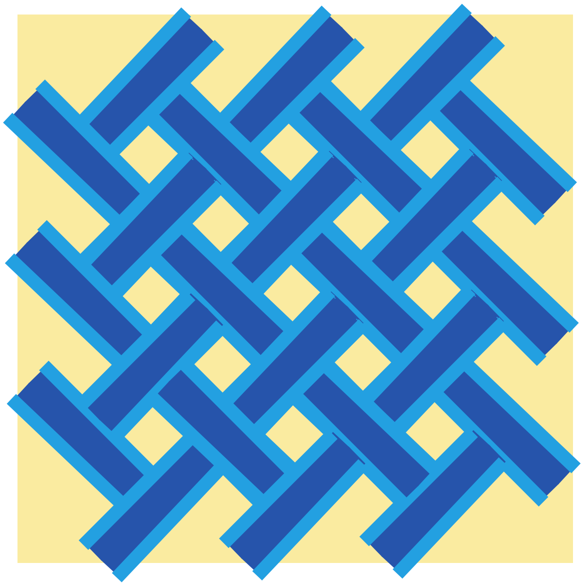


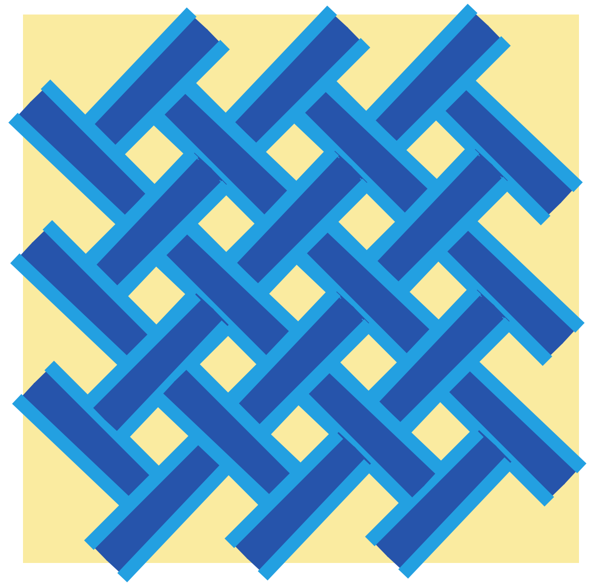
Problem 35:



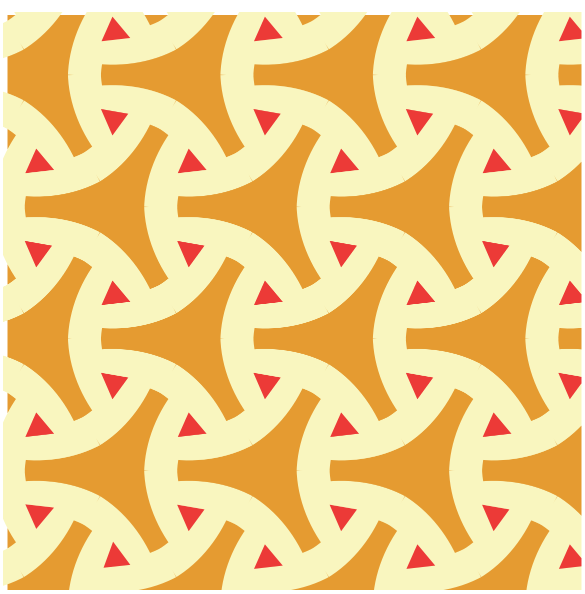


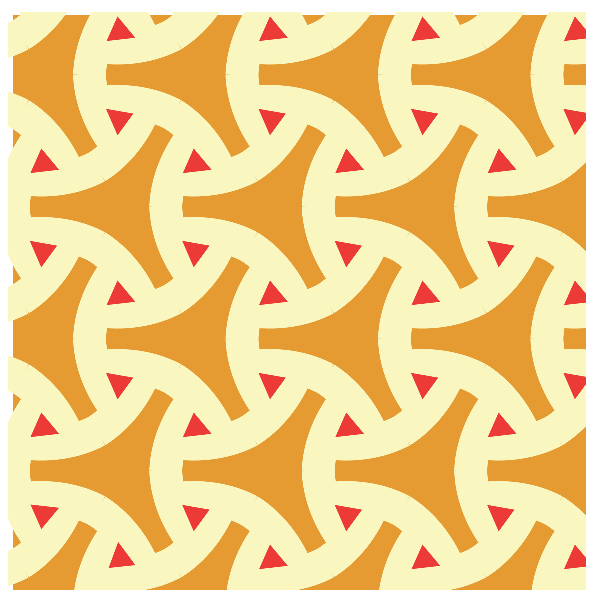
Problem 36:





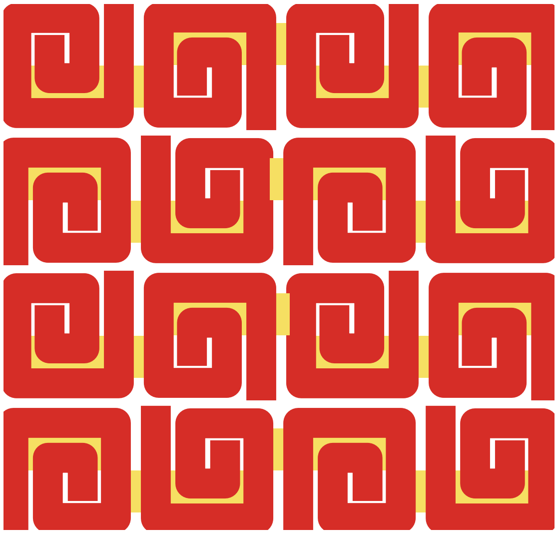
Problem 37:

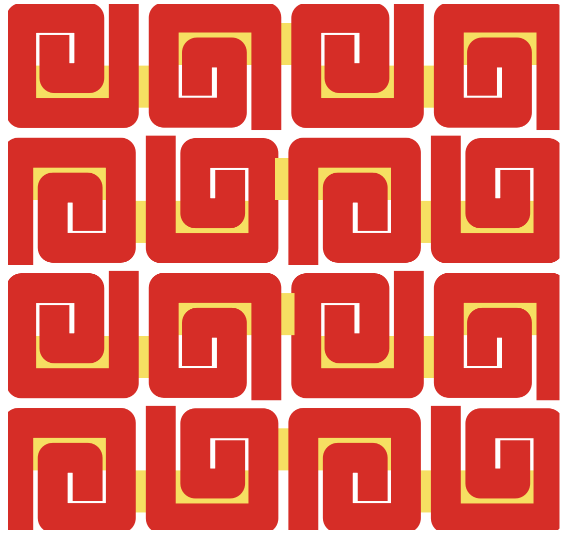




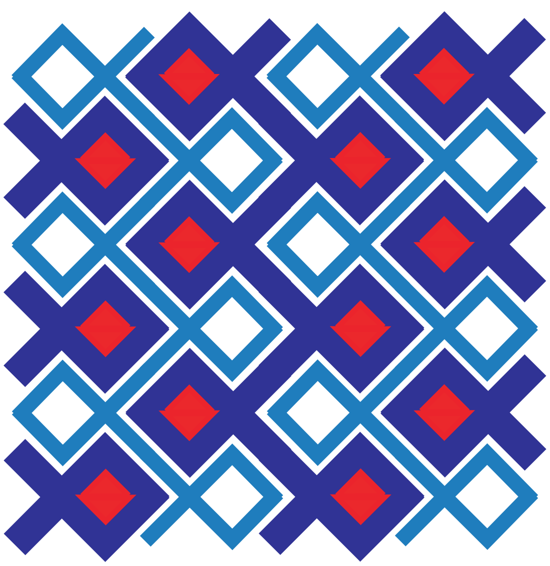
Problem 38:
Draw a wallpaper with the signature $*442$
Make sure there are no other symmetries!

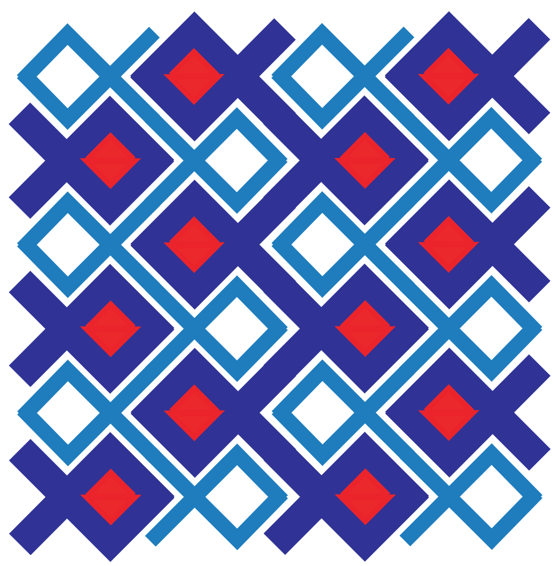
Problem 39:





Problem 40:





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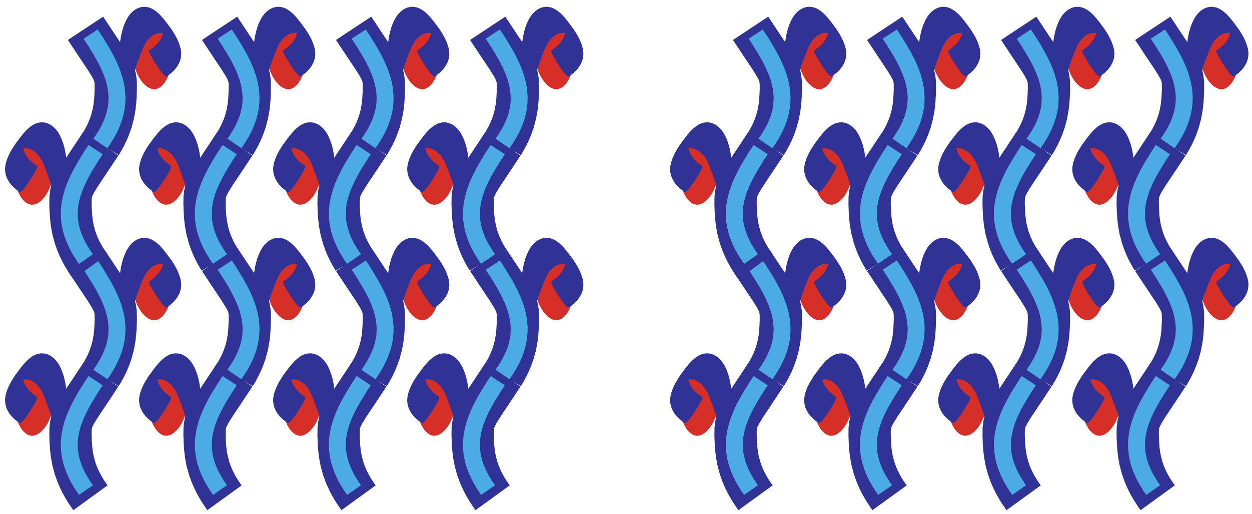
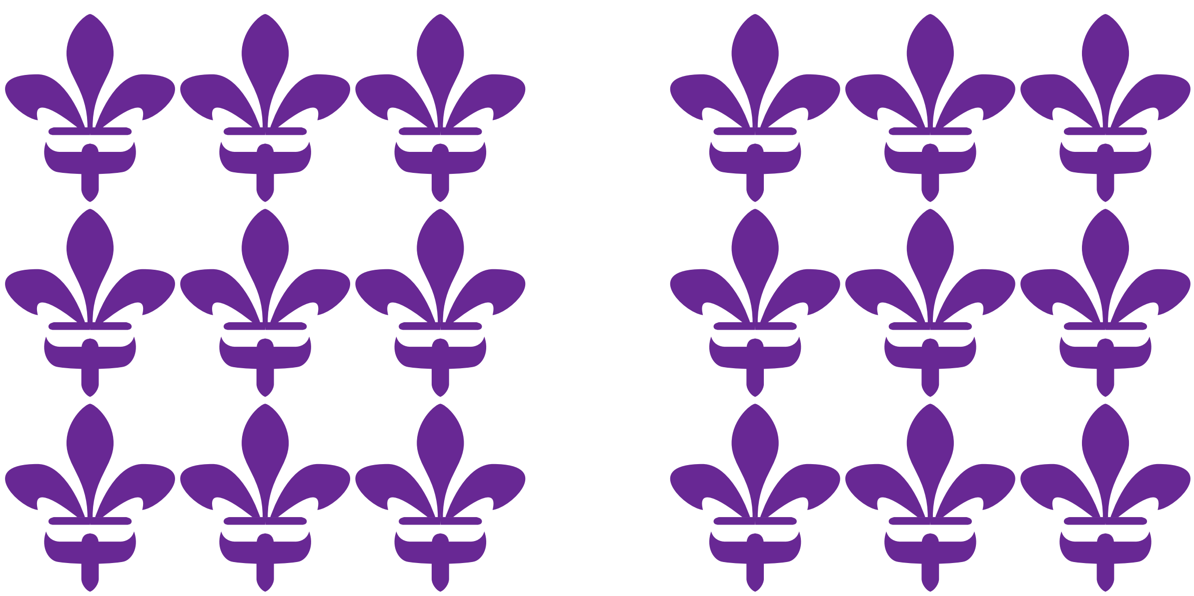
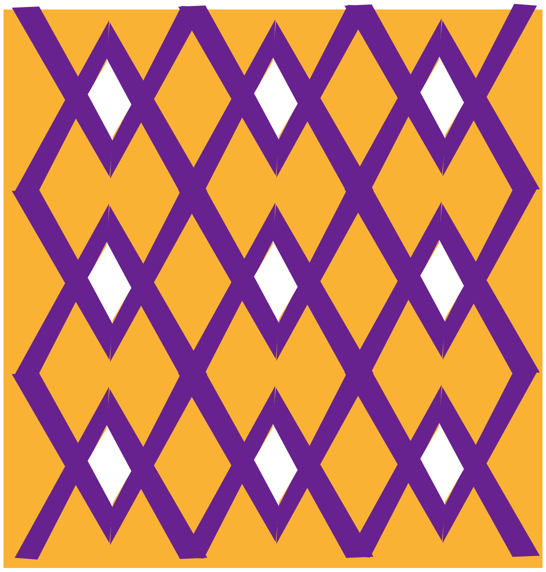
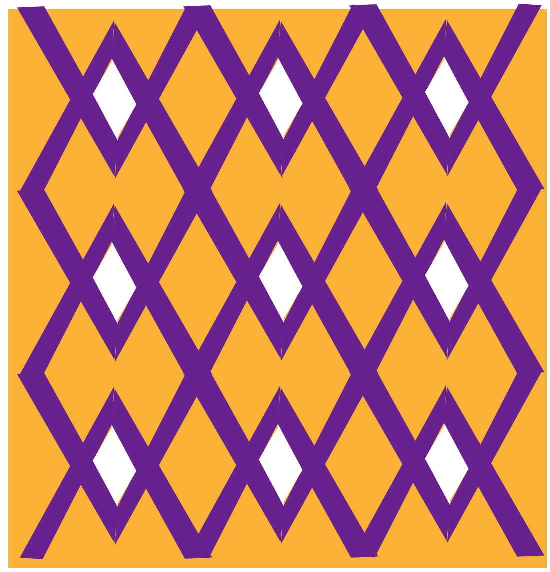


Problem 42:

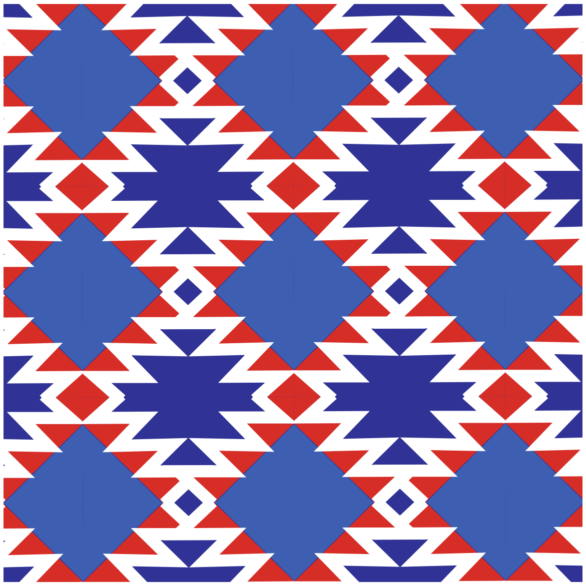


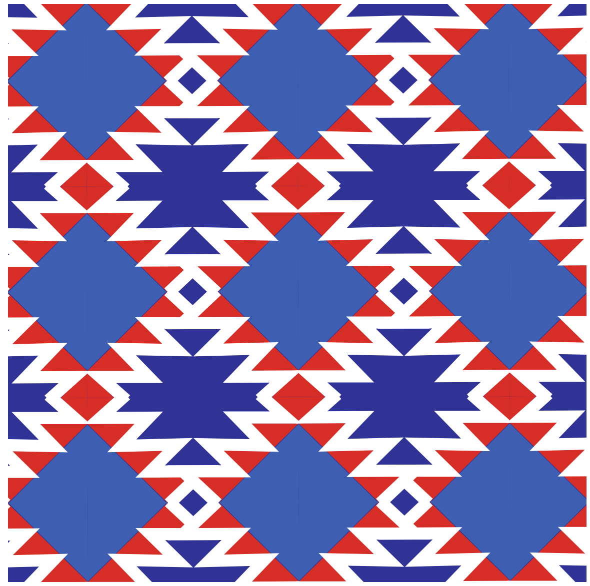
Problem 43:



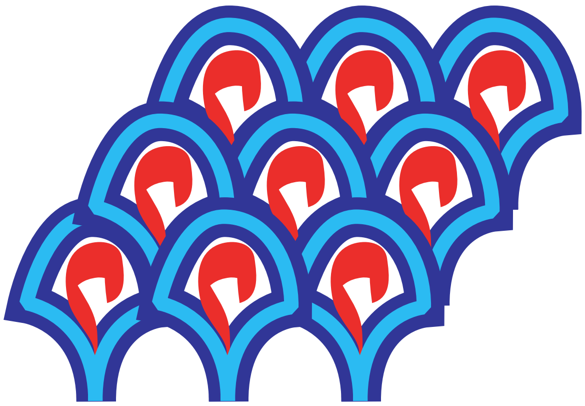


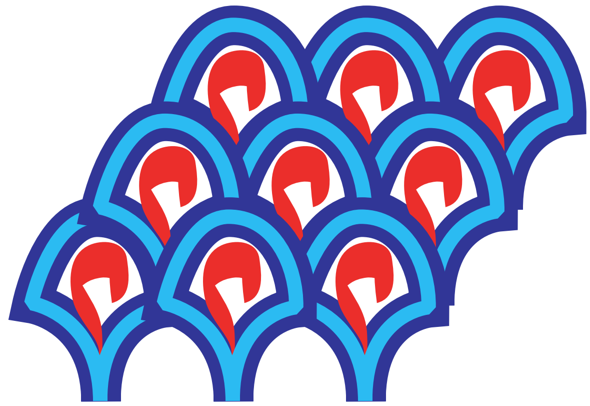
Problem 44:





Problem 45:





Section 6: The Signature-Cost Theorem

Definition 46:
First, we'll associate a *cost* to each type of symmetry in orbifold notation:

Symbol	Cost	Symbol	Cost
∘	2	× or ∗	1
⋄2	1/2	∗2	1/4
⋄3	2/3	∗3	1/3
...	...	...	...
⋄n	$\frac{n-1}{n}$	∗n	$\frac{n-1}{2n}$

We then calculate the total "cost" of a signature by adding up the costs of each component.
For example, a pattern with signature ∗333 has cost 2:

$$\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$$

Problem 47:
Calculate the costs of the following signatures:
- ⋄3∗3
- ∗∗
- ⋄4∗2:

Theorem 48:
The signatures of planar wallpaper patterns are exactly those with total cost 2.
We will not prove this theorem today, accept it without proof.

Problem 49:
Consider the 4 symmetries (translation, reflection, rotation, and glide reflection).
Which preserve orientation? Which reverse orientation?

Problem 50:
Use the signature-cost theorem to find all the signatures consisting of only ∘ or rotational symmetries.

Problem 51:
Find all the signatures consisting of only mirror symmetries.

Problem 52:
Find all the remaining signatures.
Each must be a mix of of mirror symmetries, rotational symmetries, or glide reflections.
Hint: They are all shown in the problems section.